MATRIX & DETERMINANTS

EXERCISE 1.1

Example 1.1 Find the Adjoint of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ ***and verify that*** $A(adj\ A) = (adj\ A)A = |A|I_3$

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 7 & -4 & -6 & 7 \\ -4 & 3 & 2 & -4 \\ -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 7 \end{bmatrix} \Rightarrow adjA = \begin{bmatrix} 21-16 & -8+18 & 24-14 \\ -8+18 & 24-4 & -12+32 \\ 24-14 & -12+32 & 56-36 \end{bmatrix}^T$$

$$adjA = \begin{pmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{pmatrix}^T \Rightarrow adjA = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

$$\begin{matrix} + & - & + \end{matrix}$$

$$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$|A| = 8(21-16) + 6(-18+8) + 2(24-14)$$

$$= 8(5) + 6(-10) + 2(10) = 40 - 60 + 20$$

$$= 60 - 60$$

$$\boxed{|A| = 0}$$

$$A(adj\ A) = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

$$= \begin{pmatrix} 40-60+20 & 80-120+40 & 80-120+40 \\ -30+70-40 & -60+140-80 & -60+140-80 \\ 10-40+30 & 20-80+60 & 20-80+60 \end{pmatrix}$$

$$A(adj\ A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \ldots(1)$$

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$$(adj\ A)A = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$= \begin{pmatrix} 40-60+20 & 80-120+40 & 80-120+40 \\ -30+70-40 & -60+140-80 & -60+140-80 \\ 10-40+30 & 20-80+60 & 20-80+60 \end{pmatrix}$$

$$(adj\ A)A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ldots(2)$$

$$|A|I_3 = 0\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ldots(3)$$

From (1), (2) *and* (3) $\mathrm{A(adj\ A)} = \mathrm{(adj\ A)A} = |\mathrm{A}|I$

Example 1.2 If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ***is non − singular, find*** A^{-1}

$$adjA = \begin{pmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{pmatrix}^T = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T$$

$$adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$|A| = ad - bc \neq 0$$

$$\boxed{A^{-1} = \frac{1}{|\mathrm{A}|}\mathrm{adj}(A)} \implies A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 1.3 Find the inverse of the matrices $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$

$$\boxed{A^{-1} = \frac{1}{|A|}adj(A)}$$

$$Let\ A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$$

$$\begin{matrix} + & - & + \end{matrix}$$

$$|\mathrm{A}| = \begin{vmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{vmatrix} \implies |\mathrm{A}| = 2(9-2) + 1(-15+3) + 3(-10+9)$$

$$= 2(7) + 1(-12) + 3(-1) = 14 - 12 - 3$$

$$= 14 - 15$$

$|A| = -1 \neq 0 \therefore$ *inverse exist*

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$$

$$adjA = \begin{bmatrix} \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} & (-)\begin{vmatrix} -5 & 1 \\ -3 & 3 \end{vmatrix} & \begin{vmatrix} -5 & 3 \\ -3 & 2 \end{vmatrix} \\ (-)\begin{vmatrix} -1 & 3 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -3 & 3 \end{vmatrix} & (-)\begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} & (-)\begin{vmatrix} 2 & 3 \\ -5 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ -5 & 3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 9-2 & -(-15+3) & -10+9 \\ -(-3-6) & 6+9 & -(4-3) \\ -1-9 & -(2+15) & 6-5 \end{bmatrix}^T$$

$$adjA = \begin{bmatrix} 7 & 12 & -1 \\ 9 & 15 & -1 \\ -10 & -17 & 1 \end{bmatrix}^T \Rightarrow adjA = \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix} \Rightarrow A^{-1} = -1\begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}}$$

Example 1.4 If A is a non − singular matrix of odd order, prove that adj A is positive.

Let A be a non − singular matrix of order $2m + 1$, *where* $m = 0,1,2, \ldots$

$$|A| \neq 0$$

$$|adjA| = |A|^{n-1}$$

$$|adjA| = |A|^{2m+1-1} \Rightarrow |adjA| = |A|^{2m}$$

$|A|^{2m}$ *is always positive, we get that* $|adj\ A|$ *is positive.*

Example 1.5 Find a matrix A if $adj(A) = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$

$$|adjA| = \begin{vmatrix} \overset{+}{7} & \overset{-}{7} & \overset{+}{-7} \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{vmatrix}$$

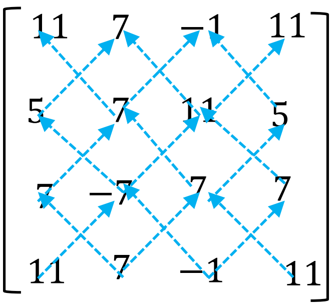
$$|adjA| = 7(77-35) - 7(-7-77) - 7(-5-121)$$

$$= 7(42) - 7(-84) - 7(-126)$$

$$= 294 + 588 + 882$$

$$= 1764 > 0$$

$$adjA = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$$



$$\Rightarrow adj(adjA) = \begin{bmatrix} 77-35 & -35-49 & 49+77 \\ 77+7 & 49+77 & 7-49 \\ -5-121 & 77-35 & 77+7 \end{bmatrix}$$

$$adj(adjA) = \begin{bmatrix} 42 & -84 & 126 \\ 84 & 126 & -42 \\ -126 & 42 & 84 \end{bmatrix}$$

$$\boxed{A = \pm\frac{1}{\sqrt{|adjA|}} adj(adjA)} \Rightarrow A = \pm\frac{1}{\sqrt{1764}} \begin{bmatrix} 42 & -84 & 126 \\ 84 & 126 & -42 \\ -126 & 42 & 84 \end{bmatrix}$$

$$A = \pm\frac{1}{42} \begin{bmatrix} 42 & -84 & 126 \\ 84 & 126 & -42 \\ -126 & 42 & 84 \end{bmatrix} = \pm \begin{bmatrix} \frac{42}{42} & -\frac{84}{42} & \frac{126}{42} \\ \frac{84}{42} & \frac{126}{42} & -\frac{42}{42} \\ -\frac{126}{42} & \frac{42}{42} & \frac{84}{42} \end{bmatrix}$$

$$A = \pm \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

Example 1.6 If $adj(A) = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, ***find*** A^{-1}

$$|adjA| = \begin{vmatrix} \overset{+}{-1} & \overset{-}{2} & \overset{+}{2} \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$

$|adjA| = -1(1-4) - 2(1-4) + 2(2-2)$

$= -1(-3) - 2(-3) + 2(0) = 3 + 6 + 0 = 9 > 0$

$|adjA| = 9$

$$\boxed{A^{-1} = \pm\frac{1}{\sqrt{|adjA|}}adjA} \Longrightarrow A^{-1} = \pm\frac{1}{\sqrt{9}}\begin{bmatrix}-1 & 2 & 2\\ 1 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$$

$$\boxed{A^{-1} = \pm\frac{1}{3}\begin{bmatrix}-1 & 2 & 2\\ 1 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}}$$

Example 1.7 If A is symmetric, prove that then adj A is also symmetric.

since A is symmetric . $A^T = A$

$(adjA)^T = adj(A^T) \Longrightarrow (adjA)^T = adjA$

$\therefore$ *adjA is symmetric .*

Example 1.8 verify the property $\boldsymbol{(A^T)^{-1} = (A^{-1})^T}$ ***with*** $\boldsymbol{A = \begin{bmatrix}2 & 9\\ 1 & 7\end{bmatrix}}$.

$A = \begin{bmatrix}2 & 9\\ 1 & 7\end{bmatrix} \Longrightarrow A^T = \begin{bmatrix}2 & 1\\ 9 & 7\end{bmatrix}$

$$\boxed{(A^T)^{-1} = \frac{1}{|A^T|}adj(A^T)}$$

$|A^T| = \begin{vmatrix}2 & 1\\ 9 & 7\end{vmatrix} = 14 - 9$

$|A| = 5 \neq 0 \therefore$ *inverse exist*

$adj(A^T) = \begin{pmatrix}7 & -1\\ -9 & 2\end{pmatrix}$

$(A^T)^{-1} = \frac{1}{5}\begin{bmatrix}7 & -1\\ -9 & 2\end{bmatrix} \dots (1)$

$A = \begin{bmatrix}2 & 9\\ 1 & 7\end{bmatrix}$

$$\boxed{A^{-1} = \frac{1}{|A|}adj(A)}$$

$|A| = \begin{vmatrix}2 & 9\\ 1 & 7\end{vmatrix} = 14 - 9$

$|A| = 5 \neq 0 \therefore$ *inverse exist*

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$$adj(A^T) = \begin{pmatrix} 7 & -9 \\ -1 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 7 & -9 \\ -1 & 2 \end{bmatrix} \Rightarrow (A^{-1})^T = \frac{1}{5}\begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix} \dots (2)$$

From (1) *and* (2) $\left(A^T\right)^{-1} = (A^{-1})^T$

Example 1.9 If $\boldsymbol{A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}}$ ***and*** $\boldsymbol{B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}}$ ***and verify that*** $\boldsymbol{(AB)^{-1} = B^{-1}A^{-1}}$

$$A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 0 & -3 \\ 1 & 4 \end{vmatrix} = 0 + 3$$

$|A| = 3 \neq 0 \therefore$ *inverse exist*

$$\text{Adj A} = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{|A|} adj\, A} \Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$$

$$|B| = \begin{vmatrix} -2 & -3 \\ 0 & -1 \end{vmatrix} = 2 - 0$$

$|B| = 2 \neq 0 \therefore$ *inverse exist*

$$Adj\, B = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$$

$$\boxed{B^{-1} = \frac{1}{|B|} adj\, B} \Rightarrow B^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 3-7 & -2+5 \\ 3-14 & -2+10 \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} \quad \dots\dots(1)$$

$$AB = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0+0 & 0+3 \\ -2+0 & -3-4 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 & 3 \\ -2 & -7 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 0 & 3 \\ -2 & -7 \end{vmatrix} = 0 + 6 = 6$$

$|AB| = 6 \neq 0 \therefore$ *inverse exist*

$\text{Adj (AB)} = \begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$

$$(AB)^{-1} = \frac{1}{|AB|}\text{adj (AB)} \implies (AB)^{-1} = \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$$

$(AB)^{-1} = \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$ ...... (2) From (1) and (2) $(AB)^{-1} = B^{-1}A^{-1}$

Example 1.10: If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 + xA + yI_2 = O_2$. Hence, find A^{-1}.

$$A^2 = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 16+6 & 12+15 \\ 8+10 & 6+25 \end{bmatrix} = \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix}$$

$$A^2 + xA + yI_2 = O_2$$

$$\begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} + x\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} + y\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} + \begin{bmatrix} 4x & 3x \\ 2x & 5x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 22+4x+y & 27+3x \\ 18+2x & 31+5x+y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$18 + 2x = 0 \implies 2x = -18 \implies x = \frac{-18}{2} \implies \boxed{x = -9}$$

Subs $x = -9$ in $22 + 4x + y = 0$

$$22 + 4(-9) + y = 0 \implies 22 - 36 + y = 0 \implies -14 + y = 0$$

$$\boxed{y = 14}$$

$$A^2 + xA + yI_2 = O_2$$

where $x = -9$ *and* $y = 14$

$$A^2 - 9A + 14I_2 = O_2$$

To find A^{-1}

$$A^2 - 9A + 14I_2 = O_2$$

Multiply by A^{-1}

$$A^2 \times A^{-1} - 9A \times A^{-1} + 14I_2 \times A^{-1} = 0$$

$$A - 9I + 14A^{-1} = 0$$

$$14A^{-1} = 9I - A \implies A^{-1} = \frac{1}{14}(9I - A)$$

$$A^{-1} = \frac{1}{14}\left(9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\right)$$

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$$= \frac{1}{14}\left(\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\right) = \frac{1}{14}\begin{bmatrix} 9-4 & 0-3 \\ 0-2 & 9-5 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{14}\begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}}$$

Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$A^TA = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$AA^T = A^TA = I_2$

A is orthogonal.

Example 1.11: Prove that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$A^TA = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$AA^T = A^TA = I_2$ $\therefore$ A is orthogonal.

Example 1.12: If $A = \frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}$ is orthogonal, find a, b and c, and hence A^{-1}.

If A is orthogonal, then $AA^T = A^TA = I_3$

$$A^T = \frac{1}{7}\begin{bmatrix} 6 & b & 2 \\ -3 & -2 & c \\ a & 6 & 3 \end{bmatrix}$$

$$AA^T = I_3$$

$$\frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix} \times \frac{1}{7}\begin{bmatrix} 6 & b & 2 \\ -3 & -2 & c \\ a & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{49}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}\begin{bmatrix} 6 & b & 2 \\ -3 & -2 & c \\ a & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}\begin{bmatrix} 6 & b & 2 \\ -3 & -2 & c \\ a & 6 & 3 \end{bmatrix} = 49\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 36 + 9 + a^2 & 6b + 6 + 6a & 12 - 3c + 3a \\ 6b + 6 + 6a & b^2 + 4 + 36 & 2b - 2c + 18 \\ 12 - 3c + 3a & 2b - 2c + 18 & 4 + c^2 + 9 \end{bmatrix} = \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$

$$\begin{bmatrix} 45 + a^2 & 6b + 6 + 6a & 12 - 3c + 3a \\ 6b + 6 + 6a & b^2 + 40 & 2b - 2c + 18 \\ 12 - 3c + 3a & 2b - 2c + 18 & c^2 + 13 \end{bmatrix} = \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$

$45 + a^2 = 49$	$b^2 + 40 = 49$	$c^2 + 13 = 49$
$a^2 = 49 - 45$	$b^2 = 49 - 40$	$c^2 = 49 - 13$
$a^2 = 4 = \sqrt{4}$	$b^2 = 9 = \sqrt{9}$	$c^2 = 36 = \sqrt{36}$
$\boxed{a = 2}$	$\boxed{b = 3}$	$\boxed{c = 6}$

1. *Find the adjoint of* (*i*) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

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$(i)\ Let\ A = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

$$adjA = \begin{pmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{pmatrix}^T = \begin{pmatrix} 2 & -6 \\ -4 & -3 \end{pmatrix}^T$$

$$adjA = \begin{bmatrix} 2 & -4 \\ -6 & -3 \end{bmatrix}$$

OR

$$Let\ A = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix} \Rightarrow Adj(A) = \begin{pmatrix} 2 & -4 \\ -6 & -3 \end{pmatrix}$$

1. *Find the adjoint of the following*: (*ii*) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$$

$$adjA = \begin{bmatrix} \begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & (-)\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ (-)\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & (-)\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & (-)\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$adjA = \begin{bmatrix} 8-7 & -(6-3) & 21-12 \\ -(6-7) & 4-3 & -(14-9) \\ 3-4 & -(2-3) & 8-9 \end{bmatrix}^T = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\boxed{adjA = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}}$$

(*iii*) $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

$$\boxed{adj(kA) = k^{n-1}adj(A)}$$

$$A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

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$$\left(\frac{1}{3}\right)^{3-1}\begin{bmatrix}1 & 2 & -2 & 1\\ -2 & 2 & 1 & -2\\ 2 & 1 & 2 & 2\\ 1 & 2 & -2 & 1\end{bmatrix} \Rightarrow \left(\frac{1}{3}\right)^{2}\begin{bmatrix}2+4 & -2-4 & 4-1\\ 2+4 & 4-1 & -2-4\\ 4-1 & 2+4 & 2+4\end{bmatrix}$$

$$adj(A) = \frac{1}{9}\begin{bmatrix}6 & -6 & 3\\ 6 & 3 & -6\\ 3 & 6 & 6\end{bmatrix} \Rightarrow \text{adj}(A) = \frac{1}{9}\times 3\begin{bmatrix}2 & -2 & 1\\ 2 & 1 & -2\\ 1 & 2 & 2\end{bmatrix}$$

$$adj(A) = \frac{1}{3}\begin{bmatrix}2 & -2 & 1\\ 2 & 1 & -2\\ 1 & 2 & 2\end{bmatrix}$$

2. *Find the inverse (if it exists)* (*i*) $\begin{bmatrix}-2 & 4\\ 1 & -3\end{bmatrix}$

(*i*) *Let* $A = \begin{bmatrix}-2 & 4\\ 1 & -3\end{bmatrix}$

$$|A| = \begin{vmatrix}-2 & 4\\ 1 & -3\end{vmatrix} = 6 - 4$$

$|A| = 2 \neq 0 \therefore$ *inverse exist*

$$\text{adj } A = \begin{bmatrix}-3 & -4\\ -1 & -2\end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{|A|}\text{adj } A} \Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix}-3 & -4\\ -1 & -2\end{bmatrix}$$

(*ii*) *Let* $A = \begin{bmatrix}5 & 1 & 1\\ 1 & 5 & 1\\ 1 & 1 & 5\end{bmatrix}$

$$|A| = \begin{vmatrix}5 & 1 & 1\\ 1 & 5 & 1\\ 1 & 1 & 5\end{vmatrix} = 5(25-1) - 1(5-1) + 1(1-5)$$

$$= 5(24) - 1(4) + 1(-4) = 120 - 4 - 4$$

$|A| = 112 \neq 0 \therefore$ *inverse exist*

$$A = \begin{bmatrix}5 & 1 & 1\\ 1 & 5 & 1\\ 1 & 1 & 5\end{bmatrix} \Rightarrow \begin{bmatrix}5 & 1 & 1 & 5\\ 1 & 5 & 1 & 1\\ 1 & 1 & 5 & 1\\ 5 & 1 & 1 & 5\end{bmatrix}$$

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$$adjA = \begin{bmatrix} 25-1 & 1-5 & 1-5 \\ 1-5 & 25-1 & 1-5 \\ 1-5 & 1-5 & 25-1 \end{bmatrix} \Rightarrow adjA = \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{|A|}\text{adj}(A)} \Rightarrow A^{-1} = \frac{1}{112}\begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{28}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}}$$

$$\textbf{(iii) Let } A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{vmatrix} = 2(8-7) - 3(6-3) + 1(21-12)$$

$$= 2(1) - 3(3) + 1(9) = 2 - 9 + 9$$

$|A| = 2 \neq 0 \therefore$ *inverse exist*

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 1 & 3 & 4 \\ 7 & 2 & 3 & 7 \\ 3 & 1 & 2 & 3 \\ 4 & 1 & 3 & 4 \end{bmatrix}$$

$$adjA = \begin{bmatrix} 8-7 & 7-6 & 3-4 \\ 3-6 & 4-3 & 3-2 \\ 21-12 & 9-14 & 8-9 \end{bmatrix} \Rightarrow adjA = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{|A|}\text{adj}(A)} \Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

3. If $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$ **show that** $[F(\alpha)]^{-1} = F(-\alpha)$

To find $[F(\alpha)]^{-1}$

$$[F(\alpha)]^{-1} = \frac{1}{|F(\alpha)|}\text{adj } F(\alpha)$$

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$$|F(\alpha)| = \begin{vmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{vmatrix} \quad \textit{Expand along } R_2$$

$$|F(\alpha)| = 1(\cos^2\alpha + \sin^2\alpha)$$

$|F(\alpha)| = 1 \neq 0 \therefore$ Inverse exist

$$\text{adj } F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{adj } F(\alpha) = \begin{bmatrix} \cos\alpha - 0 & 0 - 0 & 0 - \sin\alpha \\ 0 - 0 & \cos^2\alpha + \sin^2\alpha & 0 - 0 \\ 0 + \sin\alpha & 0 - 0 & \cos\alpha - 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$\text{adj } F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$[F(\alpha)]^{-1} = \frac{1}{1}\begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \Rightarrow [F(\alpha)]^{-1} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \ldots(1)$$

To find $F(-\alpha)$

$$F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix}$$

$$F(-\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \ldots(2)$$

From (1) and (2) $[F(\alpha)]^{-1} = F(-\alpha)$

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4. If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ show that $A^2 - 3A - 7I_2 = O_2$. Hence find A^{-1}.

$$A^2 = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$$3A = 3 \times \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix}$$

$$7I_2 = 7 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$A^2 - 3A - 7I_2 = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 22-15-7 & 9-9-0 \\ -3+3-0 & 1+6-7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^2 - 3A - 7I_2 = O_2$$

To find A^{-1}

$$A^2 - 3A - 7I_2 = 0$$

Multiply by A^{-1}

$$A^2 \times A^{-1} - 3A \times A^{-1} - 7I_2 \times A^{-1} = 0$$

$$A - 3I - 7A^{-1} = 0 \implies -7A^{-1} = 3I - A$$

$$7A^{-1} = -3I + A \implies A^{-1} = \frac{1}{7}(A - 3I)$$

$$A^{-1} = \frac{1}{7}\left(\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 3 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{7}\left(\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) = \frac{1}{7}\begin{bmatrix} 5-3 & 3-0 \\ -1-0 & -2-3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

5. If A = $\frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, prove that $A^{-1} = A^T$

$$A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj. A}$$

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$|kA| = k^n|A|$ *where n is order of A*

$$|A| = \left(\frac{1}{9}\right)^3 \begin{vmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{vmatrix}$$

$$|A| = \frac{1}{729}\begin{vmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{vmatrix} = \frac{1}{729}[-8\,(16+56) - 1(16-7) + 4(-32-\ 4)]$$

$$= \frac{1}{729}[-8(72) - 1(9) + 4(-36)] \ = \frac{1}{729}[-576 - 9 - 144]$$

$$= \frac{1}{729}[-729]$$

$|A| = -1 \neq 0 \quad \therefore$ Inverse exist

$$adj(kA) = k^{n-1}adj(A)$$

$$A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$$A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} = \left(\frac{1}{9}\right)^2 \begin{bmatrix} 4 & 7 & 4 & 4 \\ -8 & 4 & 1 & -8 \\ 1 & 4 & -8 & 1 \\ 4 & 7 & 4 & 4 \end{bmatrix}$$

$$= \left(\frac{1}{9}\right)^{3-1} \begin{bmatrix} 16+56 & -32-4 & 7-16 \\ 7-16 & -32-4 & 16+56 \\ -32-4 & 1-64 & -32-4 \end{bmatrix}$$

$$\text{adj}(A) = \frac{1}{81}\begin{bmatrix} 72 & -36 & -9 \\ -9 & -36 & 72 \\ -36 & -63 & -36 \end{bmatrix} \Longrightarrow \text{adj}(A) = \frac{1}{9\,\not{81}} \times \not{9} \begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & 8 \\ -4 & -7 & -4 \end{bmatrix}$$

$$\text{adj}(A) = \frac{1}{9}\begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & 8 \\ -4 & -7 & -4 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{|A|}\text{adj}A}$$

$$A^{-1} = \frac{1}{-1} \times \frac{1}{9}\begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & 8 \\ -4 & -7 & -4 \end{bmatrix}$$

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$$A^{-1} = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \dots (1)$$

$$A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} \Rightarrow A^T = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \quad \dots (2)$$

From (1) and (2) L.H.S = R.H.S

6. $\boldsymbol{If A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix} and\ verify\ that\ A(adj\ A) = (adj\ A)A = |A|I}$

$$Let\ A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 8 & -4 \\ -5 & 3 \end{vmatrix} = 24 - 20 = 4$$

$$\boxed{|A| = 4}$$

$$adj.\ A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$A(adj\ A) = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix} = \begin{pmatrix} 24-20 & 32-32 \\ -15+15 & -20+24 \end{pmatrix}$$

$$A(adj\ A) = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \dots .(1)$$

$$(adj\ A)A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 24-20 & -12+12 \\ 40-40 & -20+24 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(adj\ A)A = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \dots .(2)$$

$$|A|I = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \dots .(3)$$

From (1) , (2) and (3)

$$A(adj\ A) = (adj\ A)A = |A|.I$$

7. $\boldsymbol{If\ A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} and\ B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix} and\ verify\ that\ (AB)^{-1} = B^{-1}A^{-1}}$

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = 15 - 14$$

$$|A| = 1 \neq 0 \quad \therefore inverse\ exist$$

$$Adj\ (A) = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

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$A^{-1} = \frac{1}{|A|}\text{adj}A \implies A^{-1} = \frac{1}{1}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} \implies A^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

$|B| = \begin{vmatrix} -1 & -3 \\ 5 & 2 \end{vmatrix} = -2 + 15$

$|B| = 13 \neq 0 \therefore \textit{inverse exist}$

$\text{Adj } B = \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$

$B^{-1} = \frac{1}{|B|}\text{adj}B \implies B^{-1} = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$

$B^{-1} = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$= \frac{1}{13}\begin{bmatrix} 10 - 21 & -4 + 9 \\ -25 + 7 & 10 - 3 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{1}{13}\begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$ ......(1)

$(AB)^{-1} = \frac{1}{|AB|}\text{adj (AB)}$

$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix} \implies AB = \begin{bmatrix} -3 + 10 & -9 + 4 \\ -7 + 25 & -21 + 10 \end{bmatrix}$

$AB = \begin{bmatrix} 7 & -5 \\ 18 & -11 \end{bmatrix} \implies |AB| = \begin{vmatrix} 7 & -5 \\ 18 & -11 \end{vmatrix} = -77 + 90$

$|AB| = 13 \neq 0 \quad \therefore \textit{inverse exist}$

$\text{Adj (AB)} = \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{13}\begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{13}\begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$ ...... (2)

From (1) and (2) $(AB)^{-1} = B^{-1}A^{-1}$

$$\textbf{Ex : 8 If } adj(A) = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}. \textit{ find } A$$

$$|adjA| = \begin{vmatrix} \overset{+}{2} & \overset{-}{-4} & \overset{+}{2} \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{vmatrix} = 2(24-0)+4(-6-14)+2(0+24)$$

$$= 2(24)+4(-20)+2(24) = 48-80+48$$

$$|adjA| = 16 > 0$$

$$adj(adjA) = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 12 & -7 & -3 & 12 \\ 0 & 2 & -2 & 0 \\ -4 & 2 & 2 & -4 \\ 12 & -7 & -3 & 12 \end{bmatrix} \Rightarrow adj(adjA) = \begin{bmatrix} 24-0 & 0+8 & 28-24 \\ 14+6 & 4+4 & -6+14 \\ 0+24 & 8-0 & 24-12 \end{bmatrix}$$

$$adj(adjA) = \begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$\boxed{A = \pm\frac{1}{\sqrt{|adjA|}}adj(adjA)} \Rightarrow A = \pm\frac{1}{\sqrt{16}}\begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$A = \pm\frac{1}{4}\begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix} = \pm\begin{bmatrix} \frac{24}{4} & \frac{8}{4} & \frac{4}{4} \\ \frac{20}{4} & \frac{8}{4} & \frac{8}{4} \\ \frac{24}{4} & \frac{8}{4} & \frac{12}{4} \end{bmatrix}$$

$$A = \pm\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

$$\textbf{Ex 9. If } adj(A) = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}, \textit{ find } A^{-1}$$

$$|adjA| = \begin{vmatrix} \overset{+}{0} & \overset{-}{-2} & \overset{+}{0} \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{vmatrix}$$

$$|adjA| = 0(12+0) + 2(36-18) + 0(0+6)$$

$$= 0 + 2(18) + 0 = 36 > 0$$

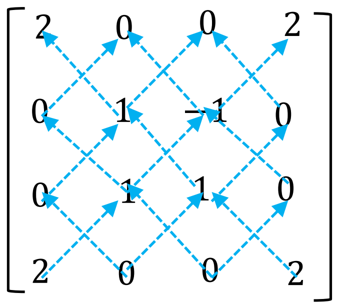
$$\boxed{|adjA| = 36}$$

$$\boxed{A^{-1} = \pm\frac{1}{\sqrt{|adjA|}}adjA} \Rightarrow A^{-1} = \pm\frac{1}{\sqrt{36}}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

$$A^{-1} = \pm\frac{1}{6}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

10. $\boldsymbol{Find\ adj(adj(A))\ if\ adj\ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}}$.

$$adj(adjA) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$



$$\Rightarrow adj(adjA) = \begin{bmatrix} 2-0 & 0-0 & 0-2 \\ 0-0 & 1+1 & 0-0 \\ 0-2 & 0-0 & 2-0 \end{bmatrix}$$

$$adj(adjA) = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

11. $\boldsymbol{A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}}$, **show that** $\boldsymbol{A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}}$.

$$A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & \tan x \\ -\tan x & 1 \end{vmatrix} = 1 - (-\tan^2 x)$$

$$\boxed{|A| = 1 + \tan^2 x}$$

$$\boxed{A^{-1} = \frac{1}{|A|}\text{adj}(A)}$$

$$\text{adj } A = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

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$$A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \times \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \frac{1}{1+\tan^2 x}\begin{bmatrix} 1-\tan^2 x & -\tan x - \tan x \\ \tan x + \tan x & -\tan^2 x + 1 \end{bmatrix}$$

$$= \frac{1}{1+\tan^2 x}\begin{bmatrix} 1-\tan^2 x & -2\tan x \\ 2\tan x & 1-\tan^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-\tan^2 x}{1+\tan^2 x} & \frac{-2\tan x}{1+\tan^2 x} \\ \frac{2\tan x}{1+\tan^2 x} & \frac{1-\tan^2 x}{1+\tan^2 x} \end{bmatrix} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$$

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A}$$

12. Find the matrix A for which $A\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$.

$$A\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$$

$$Let\ \ B = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} B^{-1}$$

$$B^{-1} = \frac{1}{|B|}\operatorname{adj}(B)$$

$$|B| = \begin{vmatrix} 5 & 3 \\ -1 & -2 \end{vmatrix} = -10 + 3$$

$|B| = -7 \neq 0$ $\therefore$ inverse exist

$$\operatorname{adj} B = \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$B^{-1} = \frac{1}{-7}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} \times \frac{1}{-7}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$A = 7\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \times \frac{1}{-7}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 4-1 & 6-5 \\ 2-1 & 3-5 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$$

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13. Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, find a matrix X such that $AXB = C$.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$AXB = C \implies XB = A^{-1}C \implies X = (A^{-1}C)B^{-1}$$

$$|A| = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 0 + 2 \implies |A| = 2 \neq 0 \quad \therefore \text{ inverse exist}$$

$$\text{adj } A = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adjA} \implies \boxed{A^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}}$$

$$|B| = \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} = 3 + 2$$

$$|B| = 5 \neq 0 \quad \therefore \text{ inverse exist}$$

$$\text{adj } B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|}\text{adjB} \implies B^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$A^{-1}C = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0+2 & 0+2 \\ -2+2 & -2+2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = \frac{1}{2} \times 2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^{-1}C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X = (A^{-1}C)B^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \times \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{5}\begin{pmatrix} 1-1 & 2+3 \\ 0+0 & 0+0 \end{pmatrix}$$

$$X = \frac{1}{5}\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} = \frac{1}{5} \times 5 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

14. If $A=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$**, show that** $A^{-1}=\frac{1}{2}(A^2-3I)$

$$A=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$$

$$\begin{matrix}+&-&+\end{matrix}$$

$$|A|=\begin{vmatrix}0&1&1\\1&0&1\\1&1&0\end{vmatrix}=0\,(0-1)-1(0-1)+1\,(1+0)$$

$$=0-1(-1)+1(1)\quad=1+1$$

$$|A|=2\neq 0$$

∴ Inverse exist

$$\begin{bmatrix}0&1&1&0\\1&0&1&1\\1&1&0&1\\0&1&1&0\end{bmatrix}\Rightarrow adjA=\begin{bmatrix}0-1&1-0&1-0\\1-0&0-1&1-0\\1-0&1-0&0-1\end{bmatrix}$$

$$\boxed{adjA=\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}}$$

$$A^{-1}=\frac{1}{|A|}(\text{adj. A})\quad\Rightarrow A^{-1}=\frac{1}{2}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}\quad\text{......(1)}$$

$$A^2=A\times A\quad\Rightarrow\quad A^2=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$$

$$A^2=\begin{bmatrix}0+1+1&0+0+1&0+1+0\\0+0+1&1+0+1&1+0+0\\0+1+0&1+0+0&1+1+0\end{bmatrix}\Rightarrow A^2=\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}$$

$$3I=3\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}$$

$$A^2-3I=\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}=\begin{bmatrix}2-3&1-0&1-0\\1-0&2-3&1-0\\1-0&1-0&2-3\end{bmatrix}=\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}$$

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$$\frac{1}{2}(A^2 - 3I) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \text{......(2)}$$

From (1) and (2) $A^{-1} = \frac{1}{2}(A^2 - 3I)$

Exercise 1.2

Example 1.13: Reduce the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ ***to a row − echelon form.***

$$\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + 2R_1} \begin{pmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 - \frac{1}{2}R_2} \begin{pmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

$R_2 + 2R_1$

−6	2	4
6	−2	4
0	0	8

$R_3 + R_1$

−3	1	2
3	−1	2
0	0	4

$R_3 - \frac{1}{2}R_2$

0	0	4
(−)	(−)	(−)
0	0	4
0	0	0

Example 1.14: Reduce the matrix $\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$ ***to a row − echelon form.***

$$\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 4 & 2 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 4R_1} \begin{pmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 2 & 8 & 20 \end{pmatrix} \xrightarrow{R_3 \to R_3 - \frac{2}{3}R_2} \begin{pmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & \frac{22}{3} & 16 \end{pmatrix}$$

$$\xrightarrow{R_3 \to 3R_3} \begin{pmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 22 & 48 \end{pmatrix}$$

$$8 - \frac{2}{3} = \frac{24-2}{3} = \frac{22}{3}$$

$R_3 + 4R_1$

4	2	0	0
−4	0	8	20
0	2	8	20

$R_3 - \frac{2}{3}R_2$

0	2	8	20
(−)	(−)	(−)	(−)
0	2	$\frac{2}{3}$	4
0	0	$\frac{22}{3}$	16

Example 1.15: Find the rank of matrices : (i) $\begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{5} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{3} & \mathbf{6} \end{bmatrix}$

$$Let\ A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$$

$A\ is\ a\ matrix\ of\ order\ 3 \times 3.$

$\therefore \rho(A) \leq min(3,3) = 3$

$The\ highest\ order\ of\ minor\ of\ A\ is\ 3.$

$$\text{It is } \begin{vmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}$$

$$= 3(6-6) - 2(6-6) + 5(3-3)$$

$$= 3(0) - 2(0) + 5(0) = 0 + 0 + 0 = 0$$

$So, \rho(A) < 3.$

$Next\ consider\ the\ second-order\ minors\ of\ A.$ $\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1 \neq 0$

$So, \rho(A) = 2.$

$(\boldsymbol{ii})$ $\begin{bmatrix} \mathbf{4} & \mathbf{3} & \mathbf{1} & \mathbf{-2} \\ \mathbf{-3} & \mathbf{-1} & \mathbf{-2} & \mathbf{4} \\ \mathbf{6} & \mathbf{7} & \mathbf{-1} & \mathbf{2} \end{bmatrix}$

$$\text{Let A} = \begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$$

$A\ is\ a\ matrix\ of\ order\ 3 \times 4.$

$\therefore \rho(A) \leq min(3,4) = 3$

$The\ highest\ order\ of\ minor\ of\ A\ is\ 3.$

$$\begin{matrix} + & - & + \end{matrix}$$
$$\begin{vmatrix} 4 & 3 & 1 \\ -3 & -1 & -2 \\ 6 & 7 & -1 \end{vmatrix} = 4\begin{vmatrix} -1 & -2 \\ 7 & -1 \end{vmatrix} - 3\begin{vmatrix} -3 & -2 \\ 6 & -1 \end{vmatrix} + 1\begin{vmatrix} -3 & -1 \\ 6 & 7 \end{vmatrix}$$

$$= 4(1+14) - 3(3+12) + 1(-21+6)$$

$$= 4(15) - 3(15) + 1(-15) = 60 - 45 - 15 = 0$$

$$\begin{matrix} + & - & + \end{matrix}$$
$$\begin{vmatrix} 4 & 3 & -2 \\ -3 & -1 & 4 \\ 6 & 7 & 2 \end{vmatrix} = 4\begin{vmatrix} -1 & 4 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} -3 & 4 \\ 6 & 2 \end{vmatrix} - 2\begin{vmatrix} -3 & -1 \\ 6 & 7 \end{vmatrix}$$

$$= 4(-2-28) - 3(-6-24) - 2(-21+6)$$

$$= 4(-30) - 3(-30) - 2(-15)$$

$$= -120 + 90 + 30 = 0$$

$$\begin{vmatrix} \overset{+}{4} & \overset{-}{1} & \overset{+}{-2} \\ -3 & -2 & 4 \\ 6 & -1 & 2 \end{vmatrix} = 4\begin{vmatrix} -2 & 4 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} -3 & 4 \\ 6 & 2 \end{vmatrix} - 2\begin{vmatrix} -3 & -2 \\ 6 & -1 \end{vmatrix}$$

$$= 4(-4+4) \quad -1(-6-24) - 2(3+12)$$

$$= 4(0) - 1(-30) - 2(15)$$

$$= 0 + 30 - 30 = 0$$

$$\begin{vmatrix} \overset{+}{3} & \overset{-}{1} & \overset{+}{-2} \\ -1 & -2 & 4 \\ 7 & -1 & 2 \end{vmatrix} = 3\begin{vmatrix} -2 & 4 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} -1 & 4 \\ 7 & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & -2 \\ 7 & -1 \end{vmatrix}$$

$$= 3(-4+4) - 1(-2-28) - 2(1+14)$$

$$= 3(0) - 1(-30) - 2(15)$$

$$= 0 + 30 - 30 = 0$$

$So, \rho(A) < 3.$

$Next\ consider\ the\ second - order\ minors\ of\ A.$

$$\begin{vmatrix} 4 & 3 \\ -3 & -1 \end{vmatrix} = -4 + 9 = 5 \neq 0$$

$So, \rho(A) = 2.$

$\boldsymbol{Example\ 1.16}$: $\boldsymbol{Find\ the\ rank\ of\ the\ following\ matrices\ which\ are\ in\ row-echelon\ form}$: $\boldsymbol{(i)} \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{-7} \\ \mathbf{0} & \mathbf{3} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$ $\boldsymbol{(ii)} \begin{bmatrix} \mathbf{-2} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{5} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$ $\boldsymbol{(iii)} \begin{bmatrix} \mathbf{6} & \mathbf{0} & \mathbf{-9} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$

$(i) \begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{Let A} = \begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$A\ is\ a\ matrix\ of\ order\ 3 \times 3.\ \therefore \rho(A) \leq 3$

$The\ third\ order\ of\ minor\ of\ |A| = \begin{vmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (2)(3)(1) = 6 \neq 0$

$\text{So}, \rho(\text{A}) = 3.$

$Note\ that\ there\ are\ three\ non-zero\ rows.$

$\boldsymbol{(ii)} \begin{bmatrix} \mathbf{-2} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{5} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \Rightarrow \text{Let A} = \begin{bmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$A\ is\ a\ matrix\ of\ order\ 3 \times 3.\ \therefore \rho(\text{A}) \leq 3$

$The\ third\ order\ of\ minor\ of\ |A| = \begin{vmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{vmatrix} = (-2)(5)(0) = 0$

$\text{So}, \rho(\text{A}) \leq 2.$

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Next consider the second − order minors of A.

$$\begin{vmatrix}-2 & 2\\ 0 & 5\end{vmatrix} = (-2)(5) = -10 \neq 0.\ \text{So}, \rho(A) = 2.$$

Note that there are two non − zero rows. The third row is a zero row.

(iii) $\begin{bmatrix}\mathbf{6} & \mathbf{0} & \mathbf{-9}\\ \mathbf{0} & \mathbf{2} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} \Rightarrow$ Let $A = \begin{bmatrix}6 & 0 & -9\\ 0 & 2 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

A is a matrix of order 4×3. $\therefore \rho(A) \leq 3$

The last two rows are zero row.

Next consider the second − order minors of A.

$$\begin{vmatrix}6 & 0\\ 0 & 2\end{vmatrix} = (6)(2) = 12 \neq 0.\ \text{So}, \rho(A) = 2.$$

Note that there are two non − zero rows. The third and fourth row is a zero row.

Example 1.17: Find the rank of the matrix $\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{3}\\ \mathbf{2} & \mathbf{1} & \mathbf{4}\\ \mathbf{3} & \mathbf{0} & \mathbf{5}\end{bmatrix}$ **by reducing it to a row − echelon form.**

$$\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 4\\ 3 & 0 & 5\end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix}1 & 2 & 3\\ 0 & -3 & -2\\ 0 & -6 & -4\end{pmatrix}$$

$R_2 - 2R_1$			$R_3 - 3R_1$		
2	1	4	3	0	5
(−)	(−)	(−)	(−)	(−)	(−)
2	4	6	3	6	9
0	−3	−2	0	−6	−4

$$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix}1 & 2 & 3\\ 0 & -3 & -2\\ 0 & 0 & 0\end{bmatrix}$$

$R_3 - 2R_2$		
0	−6	−4
(−)	(+)	(+)
0	−6	−4
0	0	0

The last equivalent matrix is in row − echelon form.

It has two non − zero rows. So, $\rho(A) = 2$.

Example 1.18: Find the rank of the matrix $\begin{bmatrix}\mathbf{2} & \mathbf{-2} & \mathbf{4} & \mathbf{3}\\ \mathbf{-3} & \mathbf{4} & \mathbf{-2} & \mathbf{-1}\\ \mathbf{6} & \mathbf{2} & \mathbf{-1} & \mathbf{7}\end{bmatrix}$ ***by reducing it to an echelon form.***

$$\begin{bmatrix}2 & -2 & 4 & 3\\ -3 & 4 & -2 & -1\\ 6 & 2 & -1 & 7\end{bmatrix} \xrightarrow{R_2 \to 2R_2} \begin{pmatrix}2 & -2 & 4 & 3\\ -6 & 8 & -4 & -2\\ 6 & 2 & -1 & 7\end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + 3R_1} \begin{pmatrix}2 & -2 & 4 & 3\\ 0 & 2 & 8 & 7\\ 0 & 8 & -13 & -2\end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 4R_2} \begin{pmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & 8 & 7 \\ 0 & 0 & -45 & -30 \end{pmatrix} \xrightarrow{R_3 \to R_3 \div (-15)} \begin{pmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & 8 & 7 \\ 0 & 0 & 3 & 2 \end{pmatrix}$$

The last equivalent matrix is in row − echelon form.

It has two non − zero rows. So, $\rho(A) = 3$.

$R_2 + 3R_1$	$R_3 - 3R_1$	$R_3 - 4R_2$
$-6 \quad 8 \quad -4 \quad -2$	$6 \quad 2 \quad -1 \quad 7$	$0 \quad 8 \quad -13 \quad -2$
	$(-) \quad (+) \quad (-) \quad (-)$	$(-) \quad (-) \quad (-) \quad (-)$
$6 \quad -6 \quad 12 \quad 9$	$6 \quad -6 \quad 12 \quad 9$	$0 \quad 8 \quad 32 \quad 28$
$0 \quad 2 \quad 8 \quad 7$	$0 \quad 8 \quad -13 \quad -2$	$0 \quad 0 \quad -45 \quad -30$

Example 1.19*: *Show that the matrix $\begin{bmatrix} \mathbf{3} & \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{5} & \mathbf{2} & \mathbf{1} \end{bmatrix}$ ***is non − singular and reduce it to the identity matrix by elementary row transformations.***

$$Let\ A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{vmatrix} \ (+\ -\ +) = 3\begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 0 \\ 5 & 2 \end{vmatrix}$$

$$= 3(0+2) - 1(2+5) + 4(4-0)$$

$$= 3(2) - 1(7) + 4(4) = 6 - 7 + 16$$

$$= 15 \neq 0$$

so, A is non − singular.

$$\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{3}R_1} \begin{pmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & -\frac{2}{3} & -\frac{11}{3} \\ 0 & \frac{1}{3} & -\frac{17}{3} \end{pmatrix} \xrightarrow{R_2 \to \left(-\frac{3}{2}\right)R_2} \begin{pmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & 1 & \frac{11}{2} \\ 0 & \frac{1}{3} & -\frac{17}{3} \end{pmatrix}$$

$R_2 - 2R_1$

$$\begin{array}{ccc} 2 & 0 & -1 \\ (-) & (-) & (-) \\ 2 & \frac{2}{3} & \frac{8}{3} \\ \hline 0 & -\frac{2}{3} & -\frac{11}{3} \end{array}$$

$R_3 - 5R_1$

$$\begin{array}{ccc} 5 & 2 & 1 \\ (-) & (-) & (-) \\ 5 & \frac{5}{3} & \frac{20}{3} \\ \hline 0 & \frac{1}{3} & -\frac{17}{3} \end{array}$$

$$-1 - \frac{8}{3} = \frac{-3-8}{3} = -\frac{11}{3}$$

$$2 - \frac{5}{3} = \frac{6-5}{3} = \frac{1}{3}$$

$$1 - \frac{20}{3} = \frac{3-20}{3} = -\frac{17}{3}$$

$$\xrightarrow[R_3 \to R_3 - \frac{1}{3}R_2]{R_1 \to R_1 - \frac{1}{3}R_2} \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{11}{2} \\ 0 & 0 & -\frac{15}{2} \end{pmatrix} \xrightarrow{R_3 \to \left(-\frac{2}{15}\right)R_3} \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{11}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$R_1 - \frac{1}{3}R_2$

$$\begin{array}{ccc} 1 & \frac{1}{3} & \frac{4}{3} \\ (-) & (-) & (-) \\ 0 & \frac{1}{3} & \frac{11}{6} \\ \hline 1 & 0 & \frac{-1}{2} \end{array}$$

$$\frac{4}{3} - \frac{11}{6} = \frac{8-11}{6} = \frac{-3}{6} = \frac{-1}{2}$$

$R_3 - \frac{1}{3}R_2$

$$\begin{array}{ccc} 0 & \frac{1}{3} & -\frac{17}{3} \\ (-) & (-) & (-) \\ 0 & \frac{1}{3} & \frac{11}{6} \\ \hline 0 & 0 & \frac{-15}{2} \end{array}$$

$$-\frac{17}{3} - \frac{11}{6} = \frac{-34-11}{6} = \frac{-45}{6} = \frac{-15}{2}$$

$$\xrightarrow[R_2 \to R_2 - \frac{11}{2}R_3]{R_1 \to R_1 + \frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_1 + \frac{1}{2}R_3$

$$\begin{array}{ccc} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ \hline 1 & 0 & 0 \end{array}$$

$R_2 - \frac{11}{2}R_3$

$$\begin{array}{ccc} 0 & 1 & \frac{11}{2} \\ (-) & (-) & (-) \\ 0 & 0 & \frac{11}{2} \\ \hline 0 & 1 & 0 \end{array}$$

1. *Find the rank of the matrices by minor method*: (*i*) $\begin{bmatrix} \mathbf{2} & \mathbf{-4} \\ \mathbf{-1} & \mathbf{2} \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

A is a matrix of order 2×2.

$\therefore \rho(A) \leq min(2,2) = 2$

The highest order of minor of A is 2.

$It\ is\ \begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix} = 4-4 = 0\ .So, \rho(A) < 2.$

$Next\ consider\ the\ minor\ of\ order\ 1$

$$|2| = 2 \neq 0\ \therefore\ \rho(A) = 1$$

$(ii) \begin{bmatrix} \mathbf{-1} & \mathbf{3} \\ \mathbf{4} & \mathbf{-7} \\ \mathbf{3} & \mathbf{-4} \end{bmatrix}$

$Let\ A = \begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}\ , A\ is\ a\ matrix\ of\ order\ 3 \times 2.$

$\therefore \rho(A) \leq min(3,2) = 2$

$The\ highest\ order\ of\ minor\ of\ A\ is\ 2.$

$It\ is\ \begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix} = 7-12 = -5\ \neq 0\ \therefore\ \rho(A) = 2$

$(iii) \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{-1} & \mathbf{0} \\ \mathbf{3} & \mathbf{-6} & \mathbf{-3} & \mathbf{1} \end{bmatrix}$

$Let\ A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

$A\ is\ a\ matrix\ of\ order\ 2 \times 4.$

$\therefore \rho(A) \leq min(2,4) = 2$

$The\ highest\ order\ of\ minor\ of\ A\ is\ 2.$

$It\ is\ \begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = -6+6 = 0$

$Also, \begin{vmatrix} -1 & 0 \\ -3 & 1 \end{vmatrix} = -1+0 = -1\ \neq 0$

$$\therefore\ \rho(A) = 2$$

$(iv) \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} & \mathbf{-6} \\ \mathbf{5} & \mathbf{1} & \mathbf{-1} \end{bmatrix} \Rightarrow Let\ A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

$A\ is\ a\ matrix\ of\ order\ 3 \times 3.$

$\therefore \rho(A) \leq min(3,3) = 3$

$The\ highest\ order\ of\ minor\ of\ A\ is\ 3.$

It is $\begin{vmatrix} \overset{+}{1} & \overset{-}{-2} & \overset{+}{3} \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 4 & -6 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -6 \\ 5 & -1 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix}$

$= 1(-4+6) + 2(-2+30) + 3\,(2-20)$

$= 1(2) + 2(28) + 3(-18) = 2 + 56 - 54 = 4 \neq 0$

$\therefore\ \rho(A) = 3.$

(v) $\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} & \mathbf{4} & \mathbf{3} \\ \mathbf{8} & \mathbf{1} & \mathbf{0} & \mathbf{2} \end{bmatrix}$

$Let\ A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$ A *is a matrix of order* 3×4.

$\therefore \rho(A) \le min(3,4) = 3$.*The highest order of minor of A is* 3.

It is $\begin{matrix} + \\ - \\ + \end{matrix}\begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 8 & 1 & 0 \end{vmatrix} = 0 + 0 + 8(4-4) = 0 + 0 + 8(0) = 0$

$\begin{vmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 4 & -6 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -6 \\ 5 & -1 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix}$ (signs $+ \ - \ +$)

$= 1(-4+6) + 2(-2+30) + 3(2-20)$

$= 1(2) + 2(28) + 3(-18) = 2 + 56 - 54 = 4 \neq 0$

$\therefore \rho(A) = 3.$

2. *Find the rank of the following matrices by row reduction method:*

(*i*) $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} & \mathbf{4} \\ \mathbf{5} & \mathbf{-1} & \mathbf{7} & \mathbf{11} \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & -4 \end{pmatrix}$

$R_2 - 2R_1$:

2	−1	3	4
(−)	(−)	(−)	(−)
2	2	2	6
0	−3	1	−2

$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$R_3 - 2R_2$:

0	−6	2	−4
(−)	(+)	(−)	(+)
0	−6	2	−4
0	0	0	0

$R_3 - 5R_1$:

5	−1	7	11
(−)	(−)	(−)	(−)
5	5	5	15
0	−6	2	−4

The last equivalent matrix is in row − echelon form.

It has two non − zero rows. So, ρ(A) = 2.

(*ii*) $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{-1} & \mathbf{2} \\ \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$

$R_4 - 3R_3$:

0	−3	2
(−)	(+)	(−)
0	−3	3
0	0	−1

$R_2 - 3R_1$:

3	−1	2
(−)	(−)	(+)
3	6	−3
0	−7	5

$R_3 \to R_3 - R_1$:

1	−2	3
(−)	(−)	(+)
1	2	−1
0	−4	4

$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_1]{R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -4 & 4 \\ 0 & -3 & 2 \end{pmatrix}$

$R_4 - R_1$:

1	−1	1
(−)	(−)	(+)
1	2	−1
0	−3	2

$(ii)\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

$R_2 - 3R_1$: $\begin{array}{ccc} 3 & -1 & 2 \\ (-) & (-) & (+) \\ 3 & 6 & -3 \\ \hline 0 & -7 & 5 \end{array}$ $R_3 - R_1$: $\begin{array}{ccc} 1 & -2 & 3 \\ (-) & (-) & (+) \\ 1 & 2 & -1 \\ \hline 0 & -4 & 4 \end{array}$

$R_4 - R_1$: $\begin{array}{ccc} 1 & -1 & 1 \\ (-) & (-) & (+) \\ 1 & 2 & -1 \\ \hline 0 & -3 & 2 \end{array}$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_1]{R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -4 & 4 \\ 0 & -3 & 2 \end{pmatrix}$$

$R_4 - 3R_3$: $\begin{array}{ccc} 0 & -3 & 2 \\ (-) & (+) & (-) \\ 0 & -3 & 3 \\ \hline 0 & 0 & -1 \end{array}$

$$\xrightarrow{R_3 \to R_3 \div 4} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -1 & 1 \\ 0 & -3 & 2 \end{pmatrix} \xrightarrow{R_4 \to R_4 - 3R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow{R_3 \to 7R_3 - R_2} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

$7R_3 - R_2$: $\begin{array}{ccc} 0 & -7 & 7 \\ (-) & (+) & (-) \\ 0 & -7 & 5 \\ \hline 0 & 0 & 2 \end{array}$ $2R_4 - R_3$: $\begin{array}{ccc} 0 & 0 & -2 \\ (-) & (-) & (-) \\ 0 & 0 & 2 \\ \hline 0 & 0 & 0 \end{array}$

$$\xrightarrow{R_4 \to 2R_4 - R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The last equivalent matrix is in row − echelon form.

It has three non − zero rows. So, $\rho(A) = 3$.

$(iii)\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

$R_2 + 2R_1$: $\begin{array}{cccc} 2 & -5 & 1 & 4 \\ -2 & 4 & 6 & -4 \\ \hline 0 & -1 & 7 & 0 \end{array}$

$$\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 4 \\ 3 & -8 & 5 & 2 \end{pmatrix}$$

$R_3 + 3R_1$: $\begin{array}{cccc} 3 & -8 & 5 & 2 \\ -3 & 6 & 9 & -6 \\ \hline 0 & -2 & 14 & -4 \end{array}$

$$\xrightarrow{R_2 \to R_2 + 2R_1,\ R_3 \to R_3 + 3R_1} \begin{pmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & -2 & 14 & -4 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 \div 2} \begin{pmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & -1 & 7 & -2 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

The last equivalent matrix is in row − echelon form.

It has three non − zero rows. So, $\rho(A) = 3$.

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Exercise 1.3

***Example 1.22**: Solve the following system of linear equations, using matrix inversion method*: $\mathbf{5x+2y=3, 3x+2y=5.}$

$5x+2y=3,\ 3x+2y=5$

$$\begin{bmatrix}5 & 2\\3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\5\end{bmatrix}$$

It is the form $AX = B$

$$A=\begin{bmatrix}5 & 2\\3 & 2\end{bmatrix}, X=\begin{bmatrix}x\\y\end{bmatrix}, B=\begin{bmatrix}3\\5\end{bmatrix}$$

$$\boxed{X=A^{-1}B}$$

To find A^{-1}: $A=\begin{bmatrix}5 & 2\\3 & 2\end{bmatrix}$

$$|A|=\begin{vmatrix}5 & 2\\3 & 2\end{vmatrix}=10-6=4\neq 0$$

$$\boxed{|A|=4}$$

$$adjA=\begin{bmatrix}2 & -2\\-3 & 5\end{bmatrix}$$

$$\boxed{A^{-1}=\frac{1}{|A|}(\text{adj A})}$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix}2 & -2\\-3 & 5\end{bmatrix}$$

$$\boxed{X=A^{-1}B}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{4}\begin{bmatrix}2 & -2\\-3 & 5\end{bmatrix}\begin{bmatrix}3\\5\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{4}\begin{bmatrix}6-10\\-9+25\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-4\\16\end{bmatrix}=\begin{bmatrix}\frac{-4}{4}\\\frac{16}{4}\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}\Rightarrow x=-1, y=4$$

***Example 1.23**: Solve the following system of equations, using matrix inversion method*: $\mathbf{2x_1+3x_2+3x_3=5, x_1-2x_2+x_3=-4, 3x_1-x_2-2x_3=3.}$

$2x_1+3x_2+3x_3=5$

$x_1-2x_2+x_3=-4$

$3x_1-x_2-2x_3=3$

$$\begin{bmatrix}2 & 3 & 3\\1 & -2 & 1\\3 & -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}5\\-4\\3\end{bmatrix}$$

It is the form $AX = B$

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$\boxed{X = A^{-1}B}$$

To find A^{-1}:

$$|A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix}$$

$$= 2(4+1) - 3(-2-3) + 3(-1+6) = 10 + 15 + 15 = 40$$

$|A| = 40 \neq 0 \therefore$ ***inverse exist***

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 & -2 \\ -1 & -2 & 3 & -1 \\ 3 & 3 & 2 & 3 \\ -2 & 1 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 4+1 & -3+6 & 3+6 \\ 3+2 & -4-9 & 3-2 \\ -1+6 & 9+2 & -4-3 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix} \qquad \boxed{A^{-1} = \frac{1}{|A|}\text{adj}(A)}$$

$$A^{-1} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\boxed{X = A^{-1}B}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} \frac{40}{40} \\ \frac{80}{40} \\ -\frac{40}{40} \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$x_1 = 1, x_2 = 2$ *and* $x_3 = -1$

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Example 1.24: ***If*** $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ ***and*** $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, ***find the products AB and BA and hence solve the system of equations*** $x - y + z = 4, x - 2y - 2z = 9, 2x + y + 3z = 1$.

$$AB = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{pmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{pmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I_3$$

$$BA = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{pmatrix} -4+7+5 & 4-1-3 & 4-3-1 \\ -4+14-10 & 4-2+6 & 4-6+2 \\ -8-7+15 & 8+1-9 & 8+3-3 \end{pmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I_3$$

$$AB = BA = 8I_3$$
$$\div 8$$

$$\frac{1}{8}AB = \frac{1}{8}BA = I_3 \Longrightarrow \left(\frac{1}{8}A\right)B = B\left(\frac{1}{8}A\right) = I_3$$

$$B\left(\frac{1}{8}A\right) = I_3 \Longrightarrow \frac{1}{8}A = B^{-1}I_3$$

$$Hence, B^{-1} = \frac{1}{8}A$$

Writing the given system of equations in matrix form, we get

$$x - y + z = 4$$
$$x - 2y - 2z = 9$$
$$2x + y + 3z = 1$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \left(\frac{1}{8}A\right)\begin{bmatrix}4\\9\\1\end{bmatrix} \Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{8}\begin{bmatrix}-4 & 4 & 4\\-7 & 1 & 3\\5 & -3 & -1\end{bmatrix}\begin{bmatrix}4\\9\\1\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{8}\begin{bmatrix}-16+36+4\\-28+9+3\\20-27-1\end{bmatrix} \Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{8}\begin{bmatrix}24\\-16\\-8\end{bmatrix}$$

$$x = 3, y = -2, z = -1$$

1. *Solve the following system of linear equations by matrix inversion method*:

$(i)\ 2x + 5y = -2, x + 2y = -3$

$$2x + 5y = -2$$

$$x + 2y = -3$$

$$\begin{bmatrix}2 & 5\\1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-2\\-3\end{bmatrix}$$

It is the form $AX = B$

$$A = \begin{bmatrix}2 & 5\\1 & 2\end{bmatrix}, X = \begin{bmatrix}x\\y\end{bmatrix}, B = \begin{bmatrix}-2\\-3\end{bmatrix}$$

$$\boxed{X = A^{-1}B}$$

To find A^{-1}: $A = \begin{bmatrix}2 & 5\\1 & 2\end{bmatrix}$

$$|A| = \begin{vmatrix}2 & 5\\1 & 2\end{vmatrix} = 4 - 5 = -1 \neq 0$$

$|A| = -1 \therefore$ ***inverse exist***

$$\text{Adj.A} = \begin{bmatrix}2 & -5\\-1 & 2\end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{|A|}(\text{adj A})} \Rightarrow A^{-1} = \frac{1}{-1}\begin{bmatrix}2 & -5\\-1 & 2\end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix}-2 & 5\\1 & -2\end{bmatrix}$$

$$\boxed{X = A^{-1}B}$$

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-2 & 5\\1 & -2\end{bmatrix}\begin{bmatrix}-2\\-3\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}4-15\\-2+6\end{bmatrix} = \begin{bmatrix}-11\\4\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-11\\4\end{bmatrix} \Rightarrow x = -11, y = 4$$

(ii) $\mathbf{2x - y = 8, 3x + 2y = -2}$

$$2x - y = 8$$

$$3x + 2y = -2$$

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$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

It is the form $AX = B$

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$$\boxed{X = A^{-1}B}$$

To find A^{-1}: $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 + 3 = 7 \neq 0$$

$|A| = 7$ $\therefore$ ***inverse exist***

$$\text{Adj.A} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

(iii) $2x + 3y - z = 9, x + y + z = 9, 3x - y - z = -1$

$$2x + 3y - z = 9$$
$$x + y + z = 9$$
$$3x - y - z = -1$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

It is the form AX = B

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$$\boxed{X = A^{-1}B}$$

To find A^{-1}:

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix} = 2(-1 + 1) - 3(-1 - 3) - 1(-1 - 3)$$

$$= 2(0) - 3(-4) - 1(-4) = 0 + 12 + 4$$

$|A| = 16 \neq 0$ $\therefore$ ***inverse exist***

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 3 & -1 \\ 3 & -1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1+1 & 1+3 & 3+1 \\ 3+1 & -2+3 & -1-2 \\ -1-3 & 9+2 & 2-3 \end{bmatrix}$$

$$\text{adj(A)} = \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix} \qquad \boxed{A^{-1} = \frac{1}{|\text{A}|}\text{adj(A)}}$$

$$A^{-1} = \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$\text{X} = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 0+36-4 \\ 36+9+3 \\ -36+99+1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 32 \\ 48 \\ 64 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \Rightarrow x = 2, y = 3 \text{ and } z = 4$$

(iv) $x + y + z - 2 = 0, 6x - 4y + 5z - 31 = 0, 5x + 2y + 2z = 13$

$$x + y + z = 2$$

$$6x - 4y + 5z = 31$$

$$5x + 2y + 2z = 13$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

It is the form $AX = B$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$\boxed{\text{X} = A^{-1}\text{B}}$$

To find A^{-1}:

$$|\text{A}| = \begin{vmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{vmatrix} = 1(-8-10) - 1(12-25) + 1(12+20)$$

$$= 1(-18) - 1(-13) + 1(32) = -18 + 13 + 32$$

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$|A| = 27 \neq 0 \therefore$ *inverse exist*

$$\begin{bmatrix} -4 & 5 & 6 & -4 \\ 2 & 2 & 5 & 2 \\ 1 & 1 & 1 & 1 \\ -4 & 5 & 6 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} -8-10 & 2-2 & 5+4 \\ 25-12 & 2-5 & 6-5 \\ 12+20 & 5-2 & -4-6 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{|A|}\text{adj}(A)}$$

$$A^{-1} = \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$\boxed{X = A^{-1}B}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}\begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{27}\begin{bmatrix} -36+0+117 \\ 26-93+13 \\ 64+93-130 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 81 \\ -54 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \Rightarrow x = 3, y = -2 \text{ and } z = 1$$

2. ***If*** $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ ***and*** $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, ***find the products AB and BA and hence solve the system of equations*** $x + y + 2z = 1, 3x + 2y + z = 7, 2x + y + 3z = 2.$

$$AB = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{pmatrix} -5+3+6 & -5+2+3 & -10+1+9 \\ 7+3-10 & 7+2-5 & 14+1-15 \\ 1-3+2 & 1-2+1 & 2-1+3 \end{pmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I_3$$

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$$BA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} -5+7+2 & 1+1-2 & 3-5+2 \\ -15+14+1 & 3+2-1 & 9-10+1 \\ -10+7+3 & 2+1-3 & 6-5+3 \end{pmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I_3$$

$$AB = BA = 4I_3$$

$$That\ is,\ \left(\frac{1}{4}A\right)B = B\left(\frac{1}{4}A\right) = I_3$$

$$Hence,\ B^{-1} = \frac{1}{4}A$$

Writing the given system of equations in matrix form, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left(\frac{1}{4}A\right)\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -5+7+6 \\ 7+7-10 \\ 1-7+2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix}$$

$$x = 2, y = 1, z = -1$$

3. *A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was Rs. 19,800 per month at the end of the first month after 3 years of service and Rs. 23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method to solve the problem.)*

Let the man's starting salary be Rs. x and his annual increment be Rs. y.

By the given data $x + 3y = 19800$

and $x + 9y = 23{,}400$.

The matrix form of the given system of equations is

$$\begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$$

It is the form $AX = B$

$$A=\begin{bmatrix}1 & 3\\ 1 & 9\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}, B=\begin{bmatrix}19800\\ 23400\end{bmatrix}$$

$$\boxed{X=A^{-1}B}$$

To find A^{-1}: $A=\begin{bmatrix}1 & 3\\ 1 & 9\end{bmatrix}$

$$|A|=\begin{vmatrix}1 & 3\\ 1 & 9\end{vmatrix}=9-3=6\neq 0$$

$$\boxed{|A|=6}$$

$$adj\ A=\begin{bmatrix}9 & -3\\ -1 & 1\end{bmatrix}$$

$$\boxed{A^{-1}=\frac{1}{|A|}(\text{adj A})}$$

$$A^{-1}=\frac{1}{6}\begin{bmatrix}9 & -3\\ -1 & 1\end{bmatrix}$$

$$\boxed{X=A^{-1}B}$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{6}\begin{bmatrix}9 & -3\\ -1 & 1\end{bmatrix}\begin{bmatrix}19800\\ 23400\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{6}\begin{bmatrix}178200-70200\\ -19800+23400\end{bmatrix}=\frac{1}{6}\begin{bmatrix}108000\\ 3600\end{bmatrix}\begin{matrix}18000\\ 600\end{matrix}$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}18000\\ 600\end{bmatrix}\Rightarrow\quad x=18000, y=600$$

Hence the man's starting is Rs.18000 and his annual increment is Rs.600.

4. 4 men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

one man alone be x days and one women alone be y days.

$$man's\ one\ day\ work=\frac{1}{x}\quad and\ woman's\ one\ day\ work=\frac{1}{y}$$

$$\therefore\frac{4}{x}+\frac{4}{y}=\frac{1}{3}\quad and\quad \frac{2}{x}+\frac{5}{y}=\frac{1}{4}$$

The matrix form of the given system of equations is

$$\begin{bmatrix}4 & 4\\ 2 & 5\end{bmatrix}\begin{bmatrix}\frac{1}{x}\\ \frac{1}{y}\end{bmatrix}=\begin{bmatrix}\frac{1}{3}\\ \frac{1}{4}\end{bmatrix}$$

It is the form $AX=B$

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$$A = \begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix}, X = \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix}, B = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$

$$\boxed{X = A^{-1}B}$$

To find A^{-1}: $A = \begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & 4 \\ 2 & 5 \end{vmatrix} = 20 - 8 = 12 \neq 0$$

$$|A| = 12$$

$$adj\ A = \begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{|A|}(\text{adj A})} \implies A^{-1} = \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$$

$$\boxed{X = A^{-1}B}$$

$$\begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} \implies \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix} = \frac{1}{12}\begin{bmatrix} \frac{5}{3} - 1 \\ \frac{-2}{3} + 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix} = \frac{1}{12}\begin{bmatrix} \frac{5-3}{3} \\ \frac{-2+3}{3} \end{bmatrix} = \frac{1}{12}\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \times \frac{1}{12} \\ \frac{1}{3} \times \frac{1}{12} \end{bmatrix} 6$$

$$\begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix} = \begin{bmatrix} \frac{1}{18} \\ \frac{1}{36} \end{bmatrix}$$

$$\therefore \frac{1}{x} = \frac{1}{18} \implies x = 18$$

$$\frac{1}{y} = \frac{1}{36} \implies y = 36$$

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5. The prices of three commodities A, B and C are Rs. x, y and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C . Person Q purchases 2 units of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 unit of B and one unit of C . In the process, P, Q and R earn Rs. 15,000, Rs. 1,000 and Rs. 4,000 respectively. Find the prices per unit of A, B and C . (Use matrix inversion method to solve the problem.)

The prices of three commodities A, B and C are Rs. x, y and z per units

$-4y + 2x + 5z = 15000 \;, -2z + 3x + y = 1000, -x + 3y + z = 4000$

$2x - 4y + 5z = 15000$

$3x + y - 2z = 1000$

$-x + 3y + z = 4000$

$$\begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

It is the form $AX = B$

$$A = \begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$\boxed{X = A^{-1}B}$$

To find A^{-1}:

$$A = \begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{vmatrix} = 2(1+6) + 4(3-2) + 5(9+1)$$

$$= 2(7) + 4(1) + 5(10) = 14 + 4 + 50$$

$|A| = 68 \neq 0 \therefore$ *inverse exist*

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 3 & 1 & -1 & 3 \\ -4 & 5 & 2 & -4 \\ 1 & -2 & 3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1+6 & 15+4 & 8-5 \\ 2-3 & 2+5 & 15+4 \\ 9+1 & 4-6 & 2+12 \end{bmatrix}$$

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$$\text{adj}(A) = \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

$$A^{-1} = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{68}\begin{bmatrix} 105000 + 19000 + 12000 \\ -15000 + 7000 + 76000 \\ 150000 - 2000 + 56000 \end{bmatrix} \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{68}\begin{bmatrix} 136000 \\ 68000 \\ 204000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2000 \\ 1000 \\ 3000 \end{bmatrix}$$

$x = 2000, y = 1000 \; and \; z = 3000$

Exercise 1.4

Example 1.25: Solve, by Cramer's rule, the system of equations $\boldsymbol{x_1 - x_2 = 3, 2x_1 + 3x_2 + 4x_3 = 17, x_2 + 2x_3 = 7}$.

$$x_1 - x_2 + 0x_3 = 3$$
$$2x_1 + 3x_2 + 4x_3 = 17$$
$$0x_1 + x_2 + 2x_3 = 7$$

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} + 0 = 1(6-4) + 1(4-0)$$

$$= 1(2) + 1(4) = 2 + 4$$

$$\Delta = 6$$

$\therefore \Delta \neq 0$, *the system has unique solution.*

$$\Delta_{x_1} = \begin{vmatrix} 3 & -1 & 0 \\ 17 & 3 & 4 \\ 7 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 17 & 4 \\ 7 & 2 \end{vmatrix} + 0 = 3(6-4) + 1(34-28)$$

$$= 3(2) + 1(6) = 6 + 6$$

$$\boxed{\Delta_{x_1} = 12}$$

$$\Delta_{x_2} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 17 & 4 \\ 0 & 7 & 2 \end{vmatrix} = 1\begin{vmatrix} 17 & 4 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} + 0 = 1(34-28) - 3(4-0)$$

$$= 1(6) - 3(4) = 6 - 12 = -6$$

$$\boxed{\Delta_{x_2} = -6}$$

$$\Delta_{x_3} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 17 \\ 0 & 1 & 7 \end{vmatrix} = 1\begin{vmatrix} 3 & 17 \\ 1 & 7 \end{vmatrix} + 1\begin{vmatrix} 2 & 17 \\ 0 & 7 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}$$

$$= 1(21-17) + 1(14-0) + 3(2-0)$$

$$= 1(4) + 1(14) + 3(2) = 4 + 14 + 6$$

$$\boxed{\Delta_{x_3} = 24}$$

$\therefore$ By Cramer's rule

$$x_1 = \frac{\Delta_{x_1}}{\Delta} \implies x_1 = \frac{12}{6}$$

$$\boxed{x_1 = 2}$$

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$$x_2 = \frac{\Delta_{x_2}}{\Delta} \implies x_2 = \frac{-6}{6}$$

$$x_2 = -1$$

$$x_3 = \frac{\Delta_{x_3}}{\Delta} \implies x_3 = \frac{24}{6}$$

$$\boxed{x_3 = 4}$$

$\therefore$ *The solution is* $x_1 = 2, x_2 = -1$ *and* $x_3 = 4$

Example 1.26: In a T20 match, Chennai Super Kings needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball traversed along a path in a vertical plane and the equation of the path is $y = ax^2 + bx + c$ ***with respect to a xy − coordinate system in the vertical plane and the ball traversed through the points (10, 8), (20, 16), (30, 18), can you conclude that Chennai Super Kings won the match? Justify your answer. (All distances are measured in metres and the meeting point of the plane of the path with the farthest boundary line is (70, 0).)***

The path $y = ax^2 + bx + c$ *passes through the points* (10, 8), (20,16), (40,22)

So, we get the system of equations

$$100a + 10b + c = 8$$
$$400a + 20b + c = 16$$
$$1600a + 40b + c = 22$$

$$\Delta = \begin{vmatrix} 100 & 10 & 1 \\ 400 & 20 & 1 \\ 1600 & 40 & 1 \end{vmatrix} = 1000 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix}$$

(signs above columns: + − +)

$$= 1000\left(1 \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} - 1 \begin{vmatrix} 4 & 1 \\ 16 & 1 \end{vmatrix} + 1 \begin{vmatrix} 4 & 2 \\ 16 & 4 \end{vmatrix} \right)$$

$$= 1000\left(1\ (2-4) - 1(4-16) + 1\ (16-32)\right)$$

$$= 1000[1(-2) - 1(-12) + 1(-16)]$$

$$= 1000[-2 + 12 - 16] = 1000[-6]$$

$$\boxed{\Delta = -6000}$$

(signs above columns: + − +)

$$\Delta_a = \begin{vmatrix} 8 & 10 & 1 \\ 16 & 20 & 1 \\ 22 & 40 & 1 \end{vmatrix} = 20 \begin{vmatrix} 4 & 1 & 1 \\ 8 & 2 & 1 \\ 11 & 4 & 1 \end{vmatrix} = 20\left(4 \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} - 1 \begin{vmatrix} 8 & 1 \\ 11 & 1 \end{vmatrix} + 1 \begin{vmatrix} 8 & 2 \\ 11 & 4 \end{vmatrix} \right)$$

$$= 20\left(4\ (2-4) - 1(8-11) + 1\ (32-22)\right) = 20[4(-2) - 1(-3) + 1(10)]$$

$$= 20[-8 + 3 + 10] = 20[5]$$

$$\boxed{\Delta_a = 100}$$

$$\Delta_b = \begin{vmatrix} 100 & 8 & 1 \\ 400 & 16 & 1 \\ 1600 & 2 & 1 \end{vmatrix} = 200 \begin{vmatrix} \overset{+}{1} & \overset{-}{4} & \overset{+}{1} \\ 4 & 8 & 1 \\ 16 & 11 & 1 \end{vmatrix}$$

$$= 200\left(1\begin{vmatrix} 8 & 1 \\ 11 & 1 \end{vmatrix} - 4\begin{vmatrix} 4 & 1 \\ 16 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 8 \\ 16 & 11 \end{vmatrix} \right)$$

$$= 200\left(1\ (8-11) - 4(4-16) + 1\,(44-128) \right)$$

$$= 200[1(-3) - 4(-12) + 1(-84)] = 200[-3 + 48 - 84]$$

$$= 200[-39]$$

$$\boxed{\Delta_b = -7800}$$

$$\Delta_c = \begin{vmatrix} 100 & 10 & 8 \\ 400 & 20 & 16 \\ 1600 & 40 & 22 \end{vmatrix}$$

$$= 2000 \begin{vmatrix} \overset{+}{1} & \overset{-}{1} & \overset{+}{4} \\ 4 & 2 & 8 \\ 16 & 4 & 11 \end{vmatrix}$$

$$= 2000\left(1\begin{vmatrix} 2 & 8 \\ 4 & 11 \end{vmatrix} - 1\begin{vmatrix} 4 & 8 \\ 16 & 11 \end{vmatrix} + 4\begin{vmatrix} 4 & 2 \\ 16 & 4 \end{vmatrix} \right)$$

$$= 2000\left(1(22-32) - 1(44-128) + 4\,(16-32) \right)$$

$$= 2000[1(-10) - 1(-84) + 4(-16)] = 2000[-10 + 84 - 64]$$

$$= 2000[10]$$

$$\boxed{\Delta_c = 20000}$$

∴ By Cramer's rule

$$a = \frac{\Delta_a}{\Delta} \Rightarrow a = \frac{100}{-6000} \Rightarrow a = -\frac{1}{60}$$

$$b = \frac{\Delta_b}{\Delta} \Rightarrow b = \frac{-7800^{13}}{-6000_{10}} \Rightarrow b = \frac{13}{10}$$

$$c = \frac{\Delta_c}{\Delta} \Rightarrow c = \frac{20000^{10}}{-6000_{-3}} \Rightarrow c = -\frac{10}{3}$$

$\therefore$ *The equation of the path is* $y = -\frac{1}{60}x^2 + \frac{13}{10}x - \frac{10}{3}$

When $x = 70$, *we get* $y = 6$. *So, the ball went by 6 metres high over the boundary line and it is impossible for a fielder standing even just before the boundary line to jump and catch the ball. Hence the ball went for a super six and the Chennai Super Kings won the match.*

1. Solve the following systems of linear equations by Cramer's rule:

(i) $\mathbf{5x - 2y + 16 = 0, x + 3y - 7 = 0}$

$$5x - 2y = -16$$

$$x + 3y = 7$$

$$\Delta = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix}$$

$$\Delta = 15 + 2 \implies \Delta = 17$$

Since $\Delta \neq 0$, *the system has unique solution*

$$\Delta_x = \begin{vmatrix} -16 & -2 \\ 7 & 3 \end{vmatrix} = -48 + 14$$

$$\Delta_x = -34$$

$$\Delta_y = \begin{vmatrix} 5 & -16 \\ 1 & 7 \end{vmatrix}$$

$$\Delta_y = 35 + 16 \implies \Delta_y = 51$$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta_x}{\Delta} \implies x = \frac{-34}{17} \implies \boxed{x = -2}$$

$$y = \frac{\Delta_y}{\Delta} \implies y = \frac{51}{17} \implies \boxed{y = 3}$$

$$\boxed{\therefore\ x = -2, y = 3}$$

(ii) $\mathbf{\frac{3}{x} + 2y = 12, \frac{2}{x} + 3y = 13}$

let $\frac{1}{x} = a$

The given equations become

$$3a + 2y = 12$$

$$2a + 3y = 13$$

$$\Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}$$

$$\Delta = 9 - 4 \implies \boxed{\Delta = 5}$$

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$Since\ \Delta \neq 0, the\ system\ has\ unique\ solution$

$$\Delta_a = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix} = 36 - 26$$

$$\boxed{\Delta_a = 10}$$

$$\Delta_y = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix}$$

$$\Delta_y = 39 - 24 \quad \Rightarrow \Delta_y = 15$$

∴ By Cramer's rule

$$a = \frac{\Delta_a}{\Delta} \Rightarrow a = \frac{10}{5} = 2 \Rightarrow \boxed{x = \frac{1}{2}}$$

$$y = \frac{\Delta_y}{\Delta} \Rightarrow y = \frac{15}{5} \Rightarrow \boxed{y = 3}$$

$$\boxed{\therefore\ x = \frac{1}{2}, y = 3}$$

2. *In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly?*
(*Use Cramer's rule to solve the problem*).

Let the number of question Answered correctly $= x$

The number of question Answered wrongly $= y$

$$\therefore x + y = 100 \dots (1)$$

Marks awarded for a Correct answer $= 1$ *and one wrong answer* $= -\frac{1}{4}$

$$\therefore (1 \times x) + \left(-\frac{1}{4} \times y\right) = 80$$

$$\therefore x - \frac{y}{4} = 80$$

Multiplying by 4,

$$4x - y = 320 \ \dots (2)$$

$$x + y = 100$$

$$4x - y = 320$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix}$$

$$\Delta = -1 - 4 \Rightarrow \Delta = -5$$

$Since\ \Delta \neq 0, the\ system\ has\ unique\ solution$

$$\Delta_x = \begin{vmatrix} 100 & 1 \\ 320 & -1 \end{vmatrix} = -100 - 320$$

$$\Delta_x = -420$$

$$\Delta_y = \begin{vmatrix} 1 & 100 \\ 4 & 320 \end{vmatrix}$$

$$\Delta_y = 320 - 400 \implies \Delta_y = -80$$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta_x}{\Delta} \implies x = \frac{-420}{-5}$$

$$\boxed{x = 84}$$

$$y = \frac{\Delta_y}{\Delta} \implies y = \frac{-80}{-5}$$

$$\boxed{y = 16}$$

$\therefore\ x = 84, y = 16$

3. *A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution? (Use Cramer's rule to solve the problem).*

Let x and y be the amount of solution containing 50% and 25% acid.

From the given data, $x + y = 10 \ldots (1)$

$50\%\ of\ x + 25\%\ of\ y = 40\%\ of\ 10$

$$\frac{50}{100}x + \frac{25}{100}y = \frac{40}{100}(10)$$

$$50x + 25y = 400$$

$\div 25$

$$2x + y = 16 \ldots (2)$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$\Delta = 1 - 2 \implies \Delta = -1$$

Since $\Delta \neq 0$, *the system has unique solution*

$$\Delta_x = \begin{vmatrix} 10 & 1 \\ 16 & 1 \end{vmatrix}$$

$$= 10 - 16 \implies \Delta_x = -6$$

$$\Delta_y = \begin{vmatrix} 1 & 10 \\ 2 & 16 \end{vmatrix} = 16 - 20 \implies \Delta_y = -4$$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta_x}{\Delta} \implies x = \frac{-6}{-1} \implies x = 6$$

$$y = \frac{\Delta_y}{\Delta} \implies y = \frac{-4}{-1} \implies y = 4$$

6 litres of solution containing 50% acid and 4 litres of solution containing 25% acid must be mixed to make 40% acid solution

4. A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself ? (Use Cramer's rule to solve the problem).

The time taken by pump A to fill the tank by itself $= x$ *minutes*

The time taken by pump B to fill the tank by itself $= y$ *minutes*

So, the part of the tank filled by pump A in 1 minute $= \frac{1}{x}$

The part of the tank filled by pump B in 1 minute $= \frac{1}{y}$

The part of the tank filled by pump A & B in 1 minute $= \frac{1}{10}$

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{1}{10}$$

If the pump B runs in reverse, then the tank will be filled by both pumps in 30 minutes.

In this case, the part of the tank filled by both pumps A & B in 1 minute

$$= \frac{1}{30}$$

$$\therefore \frac{1}{x} - \frac{1}{y} = \frac{1}{30}$$

let $\frac{1}{x} = a$ *and* $\frac{1}{y} = b$

$$a + b = \frac{1}{10} \Rightarrow 10a + 10b = 1$$

$$a - b = \frac{1}{30} \Rightarrow 30a - 30b = 1$$

$$\Delta = \begin{vmatrix} 10 & 10 \\ 30 & -30 \end{vmatrix} \implies \Delta = -600$$

$$\Delta = -300 - 300$$

Since $\Delta \neq 0$, *the system has unique solution*

$$\Delta_x = \begin{vmatrix} 1 & 10 \\ 1 & -30 \end{vmatrix}$$

$$= -30 - 10 \implies \boxed{\Delta_x = -40}$$

$$\Delta_y = \begin{vmatrix} 10 & 1 \\ 30 & 1 \end{vmatrix}$$

$\Delta_y = 10 - 30 \Rightarrow \boxed{\Delta_y = -20}$

$\therefore$ By Cramer's rule

$$a = \frac{\Delta_a}{\Delta} \Rightarrow a = \frac{-40}{-600} \Rightarrow a = \frac{1}{15}$$

$$b = \frac{\Delta_b}{\Delta} \Rightarrow b = \frac{-20}{-600} \Rightarrow b = \frac{1}{30}$$

$$a = \frac{1}{x} = \frac{1}{15} \Rightarrow \boxed{x = 15}$$

$$b = \frac{1}{y} = \frac{1}{30} \Rightarrow \boxed{y = 30}$$

Pump A will take 15 minutes to fill the tank by itself

Pump B will take 30 minutes to fill the tank by itself

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadai is Rs. 150. The cost of the two dosai, two idlies and four vadais is Rs. 200. The cost of five dosai, four idlies and two vadais is Rs. 250. The family has Rs. 350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had?

Let The cost of one dosai = Rs. x

The cost of one idly = Rs. y

The cost of one vadai = Rs. z

The given equations are

$$2x + 3y + 2z = 150$$
$$2x + 2y + 4z = 200$$
$$5x + 4y + 2z = 250$$

$$\begin{matrix} + & - & + \end{matrix}$$

$$\Delta = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & 4 \\ 5 & 4 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 5 & 4 \end{vmatrix}$$

$$= 2(4 - 16) - 3(4 - 20) + 2(8 - 10) = 2(-12) - 3(-16) + 2(-2)$$

$$= -24 + 48 - 4$$

$$\boxed{\Delta = 20}$$

$\therefore \Delta \neq 0$, *the system has unique solution.*

$$\Delta_x = \begin{vmatrix} 150 & 3 & 2 \\ 200 & 2 & 4 \\ 250 & 4 & 2 \end{vmatrix} = 150\begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 200 & 4 \\ 250 & 2 \end{vmatrix} + 2\begin{vmatrix} 200 & 2 \\ 250 & 4 \end{vmatrix}$$

$$= 150(4-16) - 3(400-1000) + 2(800-500)$$

$$= 150(-12) - 3(-600) + 2(300) = -1800 + 1800 + 600$$

$$\boxed{\Delta_x = 600}$$

$$\Delta_y = \begin{vmatrix} 2 & 150 & 2 \\ 2 & 200 & 4 \\ 5 & 250 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 200 & 4 \\ 250 & 2 \end{vmatrix} - 150\begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 200 \\ 5 & 250 \end{vmatrix}$$

$$= 2(400-1000) - 150(4-20) + 2(500-1000)$$

$$= 2(-600) - 150(-16) + 2(-500)$$

$$= -1200 + 2400 - 1000$$

$$\boxed{\Delta_y = 200}$$

$$\Delta_z = \begin{vmatrix} 2 & 3 & 150 \\ 2 & 2 & 200 \\ 5 & 4 & 250 \end{vmatrix} = 2\begin{vmatrix} 2 & 200 \\ 4 & 250 \end{vmatrix} - 3\begin{vmatrix} 2 & 200 \\ 5 & 250 \end{vmatrix} + 150\begin{vmatrix} 2 & 2 \\ 5 & 4 \end{vmatrix}$$

$$= 2(500-800) - 3(500-1000) + 150(8-10)$$

$$= 2(-300) - 3(-500) + 150(-2) = -600 + 1500 - 300$$

$$\boxed{\Delta_z = 600}$$

$\therefore$ **By Cramer's rule**

$$x = \frac{\Delta_x}{\Delta} \Rightarrow x = \frac{600}{20} \Rightarrow \boxed{x = 30}$$

$$y = \frac{\Delta_y}{\Delta} \Rightarrow y = \frac{200}{20} \Rightarrow \boxed{y = 10}$$

$$z = \frac{\Delta_z}{\Delta} \Rightarrow z = \frac{600}{20} \Rightarrow \boxed{z = 30}$$

$\therefore$ *The cost of one dosai* $= Rs.\,30$

The cost of one idly $= Rs.\,10$

The cost of one vadai $= Rs.\,30$

$The\ cost\ of\ 3\ dosai\ and\ six\ idly\ and\ six\ vadai = 3(30) + 6(10) + 6(30)$

$= 90 + 60 + 180$

$= Rs. 330$

Since the family has Rs. 350 *in hand, they able to manage to pay the bill.*

Exercise 1.5

Example 1.27: Solve the following system of linear equations, by Gaussian elimination method : $4x+3y+6z=25, x+5y+7z=13, 2x+9y+z=1$.

$4x+3y+6z=25$
$x+5y+7z=13$
$2x+9y+z=1.$

Transforming the augmented matrix to echelon form

$$\left[\begin{array}{ccc|c}4&3&6&25\\1&5&7&13\\2&9&1&1\end{array}\right]\xrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{ccc|c}1&5&7&13\\4&3&6&25\\2&9&1&1\end{array}\right)$$

$$\xrightarrow[R_3\to R_3-2R_1]{R_2\to R_2-4R_1}\left(\begin{array}{ccc|c}1&5&7&13\\0&-17&-22&-27\\0&-1&-13&-25\end{array}\right)$$

$$\xrightarrow[R_3\to R_3\div(-1)]{R_2\to R_2\div(-1)}\left(\begin{array}{ccc|c}1&5&7&13\\0&17&22&27\\0&1&13&25\end{array}\right)$$

$$\xrightarrow{R_3\to 17R_3-R_2}\left(\begin{array}{ccc|c}1&5&7&13\\0&17&22&27\\0&0&199&398\end{array}\right)$$

R_2-4R_1

4	3	6	25
(−)	(−)	(−)	(−)
4	20	28	52
0	−17	−22	−27

R_3-2R_1

2	9	1	1
(−)	(−)	(−)	(−)
2	10	14	26
0	−1	−13	−25

$17R_3-R_2$

0	17	221	425
(−)	(−)	(−)	(−)
0	17	22	27
0	0	199	398

The equivalent system is written by using the echelon form:

$x+5y+7z=13 \ldots(1)$
$17y+22z=27 \ldots(2)$
$199z=398 \ldots(3)$

From (3), we get

$$199z=398 \Rightarrow z=\frac{398}{199}\Rightarrow \boxed{z=2}$$

subs $z=2$ in (2)

$17y+22z=27 \Rightarrow 17y+22(2)=27$

$17y+44=27 \Rightarrow 17y=27-44$

$$17y=-17 \Rightarrow y=\frac{-17}{17}\Rightarrow \boxed{y=-1}$$

subs $y=-1$ and $z=2$ in (1)

$x+5y+7z=13 \Rightarrow x+5(-1)+7(2)=13$

$x-5+14=13 \Rightarrow x+9=13$

$$x = 13 - 9 \Longrightarrow \boxed{x = 4}$$

$\therefore$ *The solution is* $x = 4, y = -1$ *and* $z = 2$

Example 1.28: The upward speed $v(t)$ of a rocket at time t is approximated by $v(t) = at^2 + bt + c, 0 \le t \le 100$ where a, b and c are constants. It has been found that the speed at times $t = 3, t = 6$, and $t = 9$ seconds are respectively, 64, 133, and 208 miles per second respectively. Find the speed at time $t = 15$ seconds. (Use Gaussian elimination method.)

Since $v(3) = 64, v(6) = 133, v(9) = 208$

$$t = 3\,; v(3) = a(3)^2 + 3b + c$$
$$v(3) = 9a + 3b + c$$
$$9a + 3b + c = 64$$
$$t = 6\,; v(6) = a(6)^2 + 6b + c$$
$$v(6) = 36a + 6b + c$$
$$36a + 6b + c = 133$$
$$t = 9\,; v(9) = a(9)^2 + 9b + c$$
$$v(9) = 81a + 9b + c$$
$$81a + 9b + c = 208$$

$$9a + 3b + c = 64$$
$$36a + 6b + c = 133$$
$$81a + 9b + c = 208$$

Transforming the augmented matrix to echelon form

$$[A|B] = \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 36 & 6 & 1 & 133 \\ 81 & 9 & 1 & 208 \end{array}\right] \xrightarrow[R_3 \to R_3 - 9R_1]{R_2 \to R_2 - 4R_1} \left(\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & -6 & -3 & -123 \\ 0 & -18 & -8 & -368 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 \div (-1)]{R_2 \to R_2 \div (-3)} \left(\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 18 & 8 & 368 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 9R_2} \left(\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

$R_2 - 4R_1$

36	6	1	133
(−)	(−)	(−)	(−)
36	12	4	256
0	−6	−3	−123

$R_3 - 9R_1$

81	9	1	208
(−)	(−)	(−)	(−)
81	27	9	576
0	−18	−8	−368

$R_3 - 9R_2$

0	18	8	368
(−)	(−)	(−)	(−)
0	18	9	369
0	0	−1	−1

$$\xrightarrow{R_3 \to (-1)R_3} \left(\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Writing the equivalent equations from the row − echelon matrix, we get

$$9a + 3b + c = 64 \ldots (1)$$
$$2b + c = 41 \ldots (2)$$
$$\boxed{c = 1} \ldots (3)$$

sub $c = 1$ *in* (2)

$$2b + c = 41 \Rightarrow 2b + 1 = 41$$
$$2b = 41 - 1$$
$$2b = 40 \Rightarrow b = \frac{40}{2} \Rightarrow \boxed{b = 20}$$

subs $b = 20$ *and* $c = 1$ *in* (1) $9a + 3b + c = 64$

$$9a + 3(20) + 1 = 64$$
$$9a + 60 + 1 = 64 \Rightarrow 9a + 61 = 64$$
$$9a = 64 - 61 \Rightarrow 9a = 3 \Rightarrow a = \frac{3}{9}$$
$$\boxed{a = \frac{1}{3}}$$

$$t = 15\,; v(t) = at^2 + bt + c$$

where $a = \frac{1}{3}, b = 20$ *and* $c = 1$

$$v(15) = \frac{1}{3}(15)^2 + 20(15) + 1$$
$$v(15) = \frac{1}{3} \times 225 + 300 + 1 \quad (\text{225 and 3 cancelled: } 75)$$
$$v(15) = 75 + 301 \Rightarrow v(15) = 376$$

1. *Solve the following systems of linear equations by Gaussian elimination method*:

(*i*) $\mathbf{2x - 2y + 3z = 2, x + 2y - z = 3, 3x - y + 2z = 1}$

$$2x - 2y + 3z = 2$$
$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{array}\right)$$

$$R_2 - 2R_1: \quad \begin{array}{cccc} 2 & -2 & 3 & 2 \\ (-) & (-) & (+) & (-) \\ 2 & 4 & -2 & 6 \\ \hline 0 & -6 & 5 & -4 \end{array}$$

$$R_3 - 3R_1: \quad \begin{array}{cccc} 3 & -1 & 2 & 1 \\ (-) & (-) & (+) & (-) \\ 3 & 6 & -3 & 9 \\ \hline 0 & -7 & 5 & -8 \end{array}$$

$$6R_3 - 7R_2: \quad \begin{array}{cccc} 0 & -42 & 30 & -48 \\ (-) & (+) & (-) & (+) \\ 0 & -42 & 35 & -28 \\ \hline 0 & 0 & -5 & -20 \end{array}$$

$$\xrightarrow{R_3 \to 6R_3 - 7R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & 0 & -5 & -20 \end{array}\right)$$

The equivalent system is written by using the echelon form:

$$x + 2y - z = 3 \quad \ldots(1)$$
$$-6y + 5z = -4 \quad \ldots(2)$$
$$-5z = -20 \quad \ldots(3)$$

From $(3), -5z = -20$

$$z = \frac{-20}{-5} \Longrightarrow \boxed{z = 4}$$

subs $z = 4$ *in* $(2)\ -6y + 5z = -4$

$$-6y + 5(4) = -4 \Longrightarrow -6y + 20 = -4$$
$$-6y = -4 - 20 \Longrightarrow -6y = -24$$
$$y = \frac{-24}{-6} \Longrightarrow \boxed{y = 4}$$

subs $y = 4$ *and* $z = 4$ *in* $(1)\ x + 2y - z = 3$

$$x + 2(4) - 4 = 3 \Longrightarrow x + 8 - 4 = 3$$
$$x + 4 = 3 \Longrightarrow x = 3 - 4$$
$$\boxed{x = -1}$$

$\therefore$ *The solution is* $x = -1, y = 4$ *and* $z = 4$

(ii) $\boldsymbol{2x - 4y + 6z = 22, 3x + 8y - 5z = 27, -x + y + 2z = 2}$

$$2x - 4y + 6z = 22$$
$$3x + 8y - 5z = 27$$
$$-x + y + 2z = 2$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{array}\right)$$

$R_2 - 3R_1$			
3	8	5	27
(−)	(−)	(−)	(−)
3	6	9	33
0	2	−4	−6

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 1 & -2 & -3 \\ 0 & 3 & 5 & 13 \end{array}\right)$$

$R_3 + R_1$			
−1	1	2	2
1	2	3	11
0	3	5	13

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 11 & 22 \end{array}\right)$$

$R_3 - 3R_2$			
0	3	5	13
(−)	(−)	(+)	(+)
0	3	−6	−9
0	0	11	22

The equivalent system is written by using the echelon form:

$x + 2y + 3z = 11 \ldots (1)$

$y - 2z = -3 \ldots (2)$

$11z = 22 \ldots (3)$

From $(3)\, 11z = 22$

$z = \frac{22}{11} \Rightarrow \boxed{z = 2}$

subs $z = 2$ *in* $(2)\ y - 2z = -3$

$y - 2(2) = -3 \Rightarrow y - 4 = -3$

$y = -3 + 4 \Rightarrow \boxed{y = 1}$

subs $y = 1$ *and* $z = 2$ *in* $(1)\ x + 2y + 3z = 11$

$x + 2(1) + 3(2) = 11 \Rightarrow x + 2 + 6 = 11$

$x + 8 = 11$

$x = 11 - 8 \Rightarrow \boxed{x = 3}$

$\therefore$ *The solution is* $x = 3, y = 1$ *and* $z = 2$

2. If $ax^2 + bx + c$ is divided by $x + 3, x - 5$, and $x - 1$, the remainder are 21, 61 and 9 respectively. Find a, b and c. (Use Gaussian elimination method)

Let $P(x) = ax^2 + bx + c$

Given: $P(x)$ *divided by* $(x + 3)$ *and leaves the remainder* 21

$\therefore P(-3) = 21$

$a(-3)^2 + b(-3) + c = 21$

$9a - 3b + c = 21$

Given: $P(x)$ *divided by* $(x-5)$ *and leaves the remainder* 61

$$\therefore P(5) = 61$$
$$a(5)^2 + b(5) + c = 61$$
$$25a + 5b + c = 61$$

Given: $P(x)$ *divided by* $(x-1)$ *and leaves the remainder* 9

$$\therefore P(1) = 9$$
$$a(1)^2 + b(1) + c = 9$$
$$a + b + c = 9$$

$\therefore$ *The system of linear equations*:

$$9a - 3b + c = 21$$
$$25a + 5b + c = 61$$
$$a + b + c = 9$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 9 & -3 & 1 & 21 \\ 25 & 5 & 1 & 61 \\ 1 & 1 & 1 & 9 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 25 & 5 & 1 & 61 \\ 9 & -3 & 1 & 21 \end{array}\right)$$

$R_2 - 25R_1$

25	5	1	61
(−)	(−)	(−)	(−)
25	25	25	225
0	−20	−24	−164

$$\xrightarrow[R_3 \to R_3 - 9R_1]{R_2 \to R_2 - 25R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -20 & -24 & -164 \\ 0 & -12 & -8 & -60 \end{array}\right)$$

$R_3 - 9R_1$

9	−3	1	21
(−)	(−)	(−)	(−)
9	9	9	81
0	−12	−8	−60

$$\xrightarrow[R_3 \to \frac{1}{4}R_3]{R_2 \to \frac{1}{4}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & -3 & -2 & -15 \end{array}\right)$$

$5R_3 - 3R_2$

0	−15	−10	−75
(−)	(+)	(+)	(+)
0	−15	−18	−123
0	0	8	48

$$\xrightarrow{R_3 \to 5R_3 - 3R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & 0 & 8 & 48 \end{array}\right)$$

The equivalent system is written by using the echelon form:

$$a + b + c = 9 \quad \ldots(1)$$
$$-5b - 6c = -41 \quad \ldots(2)$$
$$8c = 48 \quad \ldots(3)$$

From $(3), 8c = 48$

$$c = \frac{48}{8} \Rightarrow \boxed{c = 6}$$

subs $c = 6$ *in* (2) $-5b - 6c = -41$

$$-5b - 6(6) = -41 \Rightarrow -5b - 36 = -41$$
$$-5b = -41 + 36 \Rightarrow -5b = -5 \Rightarrow b = \frac{-5}{-5} \Rightarrow \boxed{b = 1}$$

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$subs\ b = 1\ and\ c = 6\ in\ (1)\ a + b + c = 9$

$$a + 1 + 6 = 9 \Rightarrow a + 7 = 9$$

$$a = 9 - 7 \Rightarrow \boxed{a = 2}$$

$\therefore$ *The solution is* $a = 2, b = 1$ *and* $c = 6$

3. An amount of Rs. 65,000 is invested in three bonds at the rates of 6%, 8% and 10% per annum respectively. The total annual income is Rs. 5000. The income from the third bond is Rs. 800 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

Let the income of the first bond = Rs. x

The income of the second bond = Rs. y

The income of the third bond = Rs. z

$$x + y + z = 65000$$

$$(6\%\ of\ x) + (8\%\ of\ y) + (10\%\ of\ z) = 5000$$

$$\frac{6x}{100} + \frac{8y}{100} + \frac{10z}{100} = 5000$$

$$\frac{6x + 8y + 10z}{100} = 5000 \Rightarrow 6x + 8y + 10z = 500000 \quad \div by\ 2$$

$$3x + 4y + 5z = 250000$$

The income from the third bond is Rs. 800 more than that from the second bond $(10\%\ of\ z) - (8\%\ of\ y) = 800$

$$\frac{10z - 8y}{100} = 800 \Rightarrow -8y + 10z = 80000 \quad \div by\ 2$$

$$-4y + 5z = 40000$$

The augmented matrix is $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 3 & 4 & 5 & 250000 \\ 0 & -4 & 5 & 40000 \end{array}\right]$

$$\xrightarrow{R_2 \to R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 0 & 1 & 2 & 55000 \\ 0 & -4 & 5 & 40000 \end{array}\right)$$

$R_2 - 3R_1$			
3	4	5	250000
(−) 3	(−) 3	(−) 3	(−) 195000
0	1	2	55000

$$\xrightarrow{R_3 \to R_3 + 4R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 0 & 1 & 2 & 55000 \\ 0 & 0 & 13 & 260000 \end{array}\right)$$

$R_3 + 4R_2$			
0	−4	5	40000
0	4	8	220000
0	0	13	260000

The equivalent system is written by using the echelon form:

$$x + y + z = 65000 \quad \dots (1)$$
$$y + 2z = 55000 \quad \dots (2)$$
$$13z = 260000 \dots (3)$$

From $(3), 13z = 260000$

$$z = \frac{260000}{13} \Rightarrow \boxed{z = 20000}$$

subs $z = 20000$ *in* (2) $y + 2z = 55000$

$$y + 2(20000) = 55000 \Rightarrow y + 40000 = 55000$$

$$y = 55000 - 40000 \Rightarrow \boxed{y = 15000}$$

subs $y = 15000$ *and* $z = 20000$ *in* (1) $x + y + z = 65000$

$$x + 15000 + 20000 = 65000$$
$$x + 35000 = 65000$$
$$x = 65000 - 35000 \Rightarrow \boxed{x = 30000}$$

$\therefore$ *The price of the first bond* $= Rs. 30{,}000$

The price of the second bond $= Rs. 15{,}000$

The price of the third bond $= Rs. 20{,}000$

4. *A boy is walking along the path* $y = ax^2 + bx + c$ *through the points* $(-6, 8), (-2, -12),$ *and* $(3, 8)$. *He wants to meet his friend at* $P(7, 60)$. *Will he meet his friend?* (*Use Gaussian elimination method*)

The path $y = ax^2 + bx + c$ *passes through the points* $(-6, 8), (-2, -12),$ *and* $(3, 8)$

$(-6, 8) : y = ax^2 + bx + c$ (x y)

$$8 = a(-6)^2 + b(-6) + c \Rightarrow 36a - 6b + c = 8$$

$(-2, -12) : y = ax^2 + bx + c$ (x y)

$$-12 = a(-2)^2 + b(-2) + c \Rightarrow 4a - 2b + c = -12$$

$(3, 8) : y = ax^2 + bx + c$ (x y)

$$8 = a(3)^2 + b(3) + c \Rightarrow 9a + 3b + c = 8$$

The augmented matrix is $[A|B] = \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{array}\right]$

$$[A|B] = \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{array}\right] \xrightarrow[R_3 \to 4R_3 - R_1]{R_2 \to 9R_2 - R_1} \left(\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -12 & 8 & -116 \\ 0 & 18 & 3 & 24 \end{array}\right)$$

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$$\xrightarrow[R_3 \to \frac{1}{3}R_3]{R_2 \to \frac{1}{4}R_2} \left(\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 6 & 1 & 8 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \left(\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 0 & 5 & -50 \end{array}\right)$$

$9R_2 - R_1$

36	−18	9	−108
(−)	(+)	(−)	(−)
36	−6	1	8
0	−12	8	−116

$R_3 + 2R_2$

0	6	1	8
0	−6	4	−58
0	0	5	−50

$4R_3 - R_1$

36	12	4	32
(−)	(+)	(−)	(−)
36	−6	1	8
0	18	3	24

The equivalent system is written by using the echelon form:

$$36a - 6b + c = 8 \quad \dots(1)$$
$$-3b + 2c = -29 \quad \dots(2)$$
$$5c = -50 \quad \dots(3)$$

From $(3), 5c = -50$

$$c = \frac{-50}{5} \Longrightarrow \boxed{c = -10}$$

subs $c = -10$ *in* (2) $-3b + 2c = -29$

$$-3b + 2(-10) = -29 \Longrightarrow -3b - 20 = -29$$
$$-3b = -29 + 20 \Longrightarrow -3b = -9$$
$$b = \frac{-9}{-3} \Longrightarrow \boxed{b = 3}$$

subs $b = 3$ *and* $c = -10$ *in* (1) $36a - 6b + c = 8$

$$36a - 6(3) - 10 = 8 \Longrightarrow 36a - 18 - 10 = 8$$
$$36a - 28 = 8 \Longrightarrow 36a = 8 + 28$$
$$36a = 36 \Longrightarrow a = \frac{36}{36} \Longrightarrow \boxed{a = 1}$$

$$y = ax^2 + bx + c$$

where $a = 1, b = 3$ *and* $c = -10$

$\therefore$ *Equation of the path is* $y = x^2 + 3x - 10$

Substituting $x = 7$, *in* $y = x^2 + 3x - 10$

$$y = 7^2 + 3(7) - 10 \Longrightarrow y = 49 + 21 - 10 \Longrightarrow \boxed{y = 60}$$

Thus, the path passes through the point $P(7, 60)$.

Hence, the boy will meet his friend.

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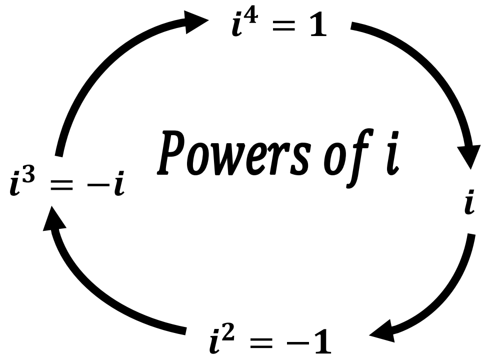
COMPLEX NUMBER

EXERCISE 2.1



Divide the exponent by 4

No remainder: *answer is* $i^0 = 1$.

Remainder of 1: *answer is* $i^1 = i$.

Remainder of 2: *answer is* $i^2 = -1$.

Remainder of 3: *answer is* $i^3 = -i$.

Sum of : $i + i^2 + i^3 + i^4 = i - 1 - i + 1 = 0$

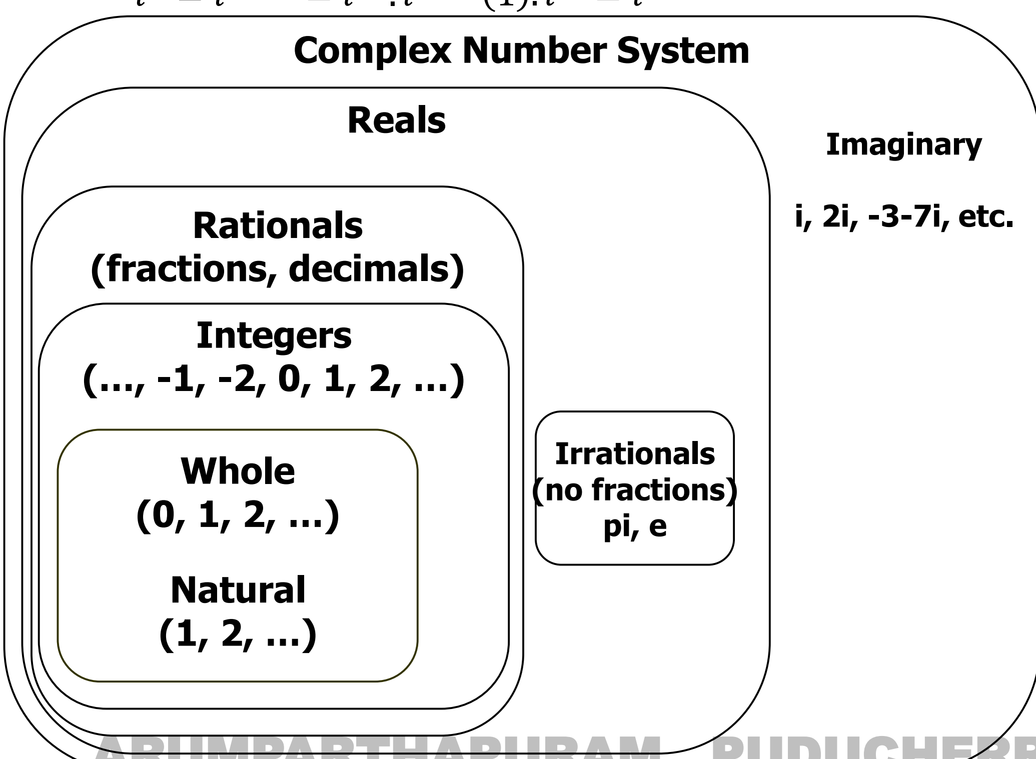
Using division algorithm find the value i^n

i^n

$n = 4q + k$

$i^n = i^{4q+k} = i^{4q}.i^k = (1).i^k = i^k$

$4\overline{)\,n\,}(q$; remainder k



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Example 2.1: ***Simplify*** i) i^7 ii) i^{729} iii) $i^{-1924} + i^{2018}$ iv) $\sum_{n=1}^{102} i^n$ v) $i\, i^2 i^3 \ldots\ldots i^{40}$

$4\overline{)7}(1$
4
3

$4\overline{)729}(182$
4
32
32
9
8
1

i) i^7

$i^7 = i^3 = -i$

ii) i^{729}

$i^{729} = i^1 = i$

$4\overline{)1924}(481$
16
32
32
4
4
0

$4\overline{)2018}(504$
20
18
16
2

iii) $i^{-1924} + i^{2018}$

$i^{-1924} + i^{2018}$

$= i^0 + i^2 = 1 + (-1)$

$= 1 - 1 = 0$

iv) $\sum_{n=1}^{102} i^n$

$= i + i^2 + i^3 + i^4 \ldots + i^{97} + i^{98} + i^{99} + i^{100} + i^{101} + i^{102}$

$= i - 1 - i + 1 + \ldots\ldots + i - 1 - i + 1 + i - 1 = -1 + i$

v) $i\, i^2 i^3 \ldots\ldots i^{40}$

$= i^{1+2+3+\ldots+40}$

$= i^{820} = i^0 = 1$

$4\overline{)820}(205$
8
20
20
0

$1 + 2 + 3 + \ldots\ldots n = \frac{n(n+1)}{2}$

$1 + 2 + 3 + \ldots\ldots 40 = \frac{40(40+1)}{2}$ (cancelling 40 and 2 gives 20)

$= 20(41) = 820$

Simply the following :

1. $i^{1947} + i^{1950}$

$= i^{1947} + i^{1950} = i^3 + i^2$

$= -i - 1 = -(i+1)$

2) $i^{1948} - i^{-1869}$

$= i^{1948} - i^{-1869}$

$= i^0 - i^{-1} = 1 - \frac{1}{i}$

$= 1 - \frac{1}{i} \times \frac{i}{i} = 1 - \frac{i}{i^2}$

$= 1 - \frac{i}{(-1)} = 1 + i$

$4\overline{)1948}(487$: 16, 34, 32, 28, 28, 0

$4\overline{)1947}(486$: 16, 34, 32, 27, 24, 3

$4\overline{)1950}(487$: 16, 35, 32, 30, 28, 2

$4\overline{)1869}(467$: 16, 26, 24, 29, 28, 1

3. $\sum_{n=1}^{12} i^n$

$$\sum_{n=1}^{12} i^n = i + i^2 + i^3 + i^4 + \ldots\ldots i^9 + i^{10} + i^{11} + i^{12}$$

$$= i - 1 - i + 1 + \ldots\ldots + i - 1 - i + 1$$

$$\sum_{n=1}^{12} i^n = 0$$

4. $i^{59} + \frac{1}{i^{59}}$

$$i^{59} + \frac{1}{i^{59}} = i^3 + \frac{1}{i^3} = -i + \frac{1}{-i} = -i - \frac{1}{i}$$

$$= -i - \frac{1}{i} \times \frac{i}{i} = -i - \frac{i}{i^2}$$

$$= -i - \frac{i}{-1} = -i + i = 0$$

4 | 59 = 14
4
19
16
3

5. $i\, i^2 i^3 \ldots\ldots i^{2000}$

$$i\, i^2 i^3 \ldots\ldots i^{2000} = i^{1+2+3+\cdots+2000}$$

$$= i^{200100} = i^0 = 1$$

$$1 + 2 + 3 + 4 + \ldots\ldots 2000 = \frac{\overset{1000}{2000}(2000+1)}{2}$$

$$= 1000(2001)$$

$$= 2001000$$

4 | 2001000 = 500250
200
10
8
20
20
0

6. $\sum_{n=1}^{10} i^{n+50}$

$$= i^{1+50} + i^{2+50} + i^{3+50} + i^{4+50} + i^{5+50} + i^{6+50} + i^{7+50} + i^{8+50} + i^{9+50} + i^{10+50}$$

$$= i^{51} + i^{52} + i^{53} + i^{54} + i^{55} + i^{56} + i^{57} + i^{58} + i^{59} + i^{60}$$

$$= i^3 + i^0 + i + i^2 + i^3 + i^0 + i + i^2 + i^3 + i^0$$

$$= -i + 1 + i - 1 - i + 1 + i - 1 - i + 1 = -i + 1$$

$$= 1 - i$$

EXERCISE 2.2

Example 2.2: Find the values of the real numbers x and y if the complex number $(2+i)x+(1-i)y+2i-3$ and $x+(-1+2i)y+1+i$ are equal.

Let $z_1 = (2+i)x+(1-i)y+2i-3$

$= 2x+ix+y-iy+2i-3 = 2x+y-3+ix-iy+2i$

$z_1 = 2x+y-3+i(x-y+2)$

$z_2 = x+(-1+2i)y+1+i = x-y+i2y+1+i$

$z_2 = x-y+1+i2y+i$

$z_2 = x-y+1+i(2y+1)$

$\boxed{z_1 = z_2}$

$2x+y-3+i(x-y+2) = x-y+1+i(2y+1)$

Equating real and imaginary part

$2x+y-3 = x-y+1, \quad x-y+2 = 2y+1$

$2x+y-3-x+y-1 = 0, \quad x-y+2-2y-1 = 0$

$x+2y-4 = 0 \ldots (1) \quad x-3y+1 = 0 \ldots (2)$

Solving (1) *and* (2)

$(1) \Rightarrow x+2y-4 = 0$

$\quad (-) \; (+) \quad (-)$

$(2) \Rightarrow x-3y+1 = 0$

$5y-5 = 0 \Rightarrow 5y = 5 \Rightarrow y = \frac{5}{5} \Rightarrow \boxed{y = 1}$

Sub $y = 1$ *in* (1) $x+2y-4 = 0$

$x+2(1)-4 = 0 \Rightarrow x+2-4 = 0$

$x-2 = 0 \Rightarrow \boxed{x = 2}$

$\therefore x = 2, y = 1$

1. *Evaluate the following if $z = 5-2i$ and $w = -1+3i$*

i) $z+w$ ii) $z-iw$ iii) $2z+3w$ iv) zw v) $z^2+2zw+w^2$ vi) $(z+w)^2$

Given: $z = 5-2i, w = -1+3i$

i) $z+w$

$z+w = 5-2i+(-1+3i) = 5-2i-1+3i = 4+i$

ii) $z-iw$

$z-iw = 5-2i-i(-1+3i)$

$= 5-2i+i-3i^2 = 5-2i+i-3(-1) = 5-2i+i+3$

$= 8-i$

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iii) $\boldsymbol{2z+3w}$

$= 2(5-2i)+3(-1+3i) = 10-4i-3+9i$

$= 7+5i$

iv) $\boldsymbol{zw}$

$zw = (5-2i)(-1+3i) = -5+15i+2i-6i^2$

$= -5+17i-6\,(-1) \; = -5+17i+6$

$= 1+17i$

v) $\boldsymbol{z^2+2zw+w^2}$

$= (5-2i)^2+2(5-2i)(-1+3i)+(-1+3i)^2$

$= 5^2+(2i)^2-2(5)(2i)+2(1+17i)+(-1)^2+(3i)^2+2(-1)(3i)$

$= 25+4\overset{(-1)}{i^2}-20i+2+34i+1+9\overset{(-1)}{i^2}-6i$

$(5-2i)(-1+3i) = 1+17i$

$= 25-4-20i+2+34i+1-9-6i$

$= 28-13+34i-26i \; = 15+8i$

vi) $\boldsymbol{(z+w)^2}$

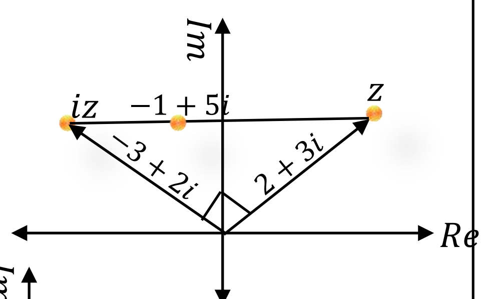
$= [5-2i-1+3i]^2$

$= (4+i)^2 \; = 4^2+2(i)(4)+i^2$

$= 16+8i-1 \; = 15+8i$

2. ***Given the complex number*** $\boldsymbol{z=2+3i}$, ***represent the complex numbers in Argand diagram. i)*** $\boldsymbol{z, iz}$ ***and*** $\boldsymbol{z+iz}$, ***ii)*** $\boldsymbol{z, -iz}$ ***and*** $\boldsymbol{z-iz}$

i) $\boldsymbol{z, iz}$ ***and*** $\boldsymbol{z+iz}$

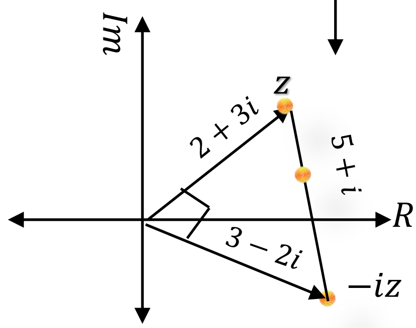


Given: $z = 2+3i$

$iz = i(2+3i) = 2i+3\overset{(-1)}{i^2} \; = -3+2i$

$z+iz = \; 2+3i-3+2i = -1+5i$

ii) $\boldsymbol{z, -iz}$ ***and*** $\boldsymbol{z-iz}$



Given: $z = 2+3i$

$-iz = -i(2+3i)$

$= -2i-3i^2 = -2i+3 = 3-2i$

$z+(-iz) = \; 2+3i+3-2i \; = 5+i$

3. ***Find the value of the real number x and y if the complex numbers*** $\boldsymbol{(3-i)x-(2-i)y+2i+5}$ ***and*** $\boldsymbol{2x+(-1+2i)y+3+2i}$ ***are equal.***

Let $z_1 = (3-i)x-(2-i)y+2i+5$

$= 3x-ix-(2y-iy)+2i+5$

$= 3x - ix - 2y + iy + 2i + 5 = 3x - 2y + 5 + i(-x + y + 2)$

$z_2 = 2x + (-1 + 2i)y + 3 + 2i$

$= 2x - y + i2y + 3 + 2i = 2x - y + 3 + i(2y + 2)$

$Given\ z_1 = z_2$

$3x - 2y + 5 + i(-x + y + 2) = 2x - y + 3 + i(2y + 2)$

Equating real and imaginary part

$3x - 2y + 5 = 2x - y + 3, \ -x + y + 2 = 2y + 2$

$3x - 2y + 5 - 2x + y - 3 = 0, -x + y + 2 - 2y - 2 = 0$

$x - y + 2 = 0 \qquad -x - y = 0 \dots (2)$

$x - y = -2 \dots (1)$

Solving (1) *and* (2)

$(1) \Rightarrow \quad x - y = -2$

$(2) \Rightarrow -x - y = 0$

$-2y = -2 \Rightarrow y = \frac{-2}{-2}$

$\boxed{y = 1}$

Sub $y = 1$ *in* (1) $x - y = -2$

$x - 1 = -2 \Rightarrow x = -2 + 1 \Rightarrow \boxed{x = -1}$

$\therefore x = -1, y = 1$

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EXERCISE 2.3

Properties of Complex number

1. *Commutative Property under addition*:

$$z_1 + z_2 = z_2 + z_1$$

Under multiplication

$$z_1 z_2 = z_2 z_1$$

2. *Associative property under addition*:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

Under multiplication

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

3. *Additive inverse*:

$$z + (-z) = -z + z = 0$$

Multiplicative inverse of w & z.

$$zw = wz = 1$$

$$\boxed{w = \frac{1}{z}}$$

4. *Multiplication distributes over addition*:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

1. $\boldsymbol{If\ z_1 = 1 - 3i, z_2 = -4i\ and\ z_3 = 5\ Show\ that}$

$\boldsymbol{i)\ (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)}$

$\boldsymbol{ii)\ (z_1 z_2) z_3 = z_1 (z_2 z_3)}$

Given: $z_1 = 1 - 3i, z_2 = -4i, z_3 = 5$

$i)\ (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

L.H.S

$$(Z_1 + Z_2) + Z_3 = [1 - 3i - 4i] + 5$$

$$= 1 - 7i + 5 = 6 - 7i$$

R.H.S

$$z_1 + (z_2 + z_3) = 1 - 3i + [-4i + 5]$$

$$= 1 - 3i - 4i + 5 = 6 - 7i$$

$$L.H.S = R.H.S$$

$\boldsymbol{ii)\ (z_1 z_2) z_3 = z_1 (z_2 z_3)}$

L.H.S

$$(z_1 z_2) z_3 = [(1 - 3i)(-4i)]\ (5)$$

$$= (-4i + 12i^2)\ (5)$$

$$= (-4i + 12(-1))(5)$$

$$= (-4i - 12)(5) = -20i - 60$$

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$R.H.S$

$z_1(z_2z_3) = (1-3i)[(-4i)(5)] = (1-3i)(-20i) = -20i + 60i^2$

$= -20i + (-60) = -20i - 60$

$L.H.S = R.H.S$

2. If $z_1 = 3, z_2 = -7i$ and $z_3 = 5 + 4i$ Show that
i) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$
ii) $(z_1 + z_2)z_3 = z_1z_3 + z_2z_3$

$Given: z_1 = 3, z_2 = -7i, z_3 = 5 + 4i$

i) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

$L.H.S$

$z_1(z_2 + z_3) = 3(-7i + 5 + 4i)$

$= 3(5 - 3i) = 15 - 9i$

$R.H.S$

$z_1z_2 + z_1z_3 = (3)(-7i) + 3(5 + 4i)$

$= -21i + 15 + 12i = 15 - 9i$

$L.H.S = R.H.S$

ii) $(z_1 + z_2)z_3 = z_1z_3 + z_2z_3$

$L.H.S$

$(z_1 + z_2)z_3 = [3 - 7i][5 + 4i]$

$= 15 + 12i - 35i - 28i^2$

$= 15 - 23i - 28(-1)$

$= 15 - 23i + 28 = 43 - 23i$

$R.H.S$

$z_1z_3 + z_2z_3 = 3(5 + 4i) + (-7i)(5 + 4i)$

$= 15 + 12i - 35i - 28i^2$

$= 15 - 23i - 28(-1)$

$= 15 - 23i + 28 = 43 - 23i$

$L.H.S = R.H.S$

3. If $z_1 = 2 + 5i, z_2 = -3 - 4i, z_3 = 1 + i$ find the additive and multiplicative inverse of $z_1 z_2$ and z_3

$(a + ib)(a - ib) = a^2 + b^2$

$z_1 = 2 + 5i$

Additive inverse of z_1 *is* $-2 - 5i$

Multiplicative inverse of z_1 *is* $\frac{1}{z_1} = \frac{1}{2+5i} = \frac{(2-5i)}{(2+5i)(2-5i)}$

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$$= \frac{2-5i}{2^2+5^2} = \frac{2-5i}{4+25} = \frac{2-5i}{29}$$

$$\boxed{z_2 = -3-4i}$$

Additive inverse of z_2 *is* $3+4i$

Multiplicative inverse of z_2 *is* $\frac{1}{z_2} = \frac{1}{-3-4i} = \frac{1}{-(3+4i)}$

$$= \frac{-1}{3+4i} \times \frac{3-4i}{3-4i} = \frac{-1(3-4i)}{(3+4i)(3-4i)}$$

$$= \frac{-3+4i}{3^2+4^2} = \frac{-3+4i}{9+16} = \frac{-3+4i}{25}$$

$z_3 = 1+i$

Additive inverse of z_3 *is* $-1-i$.

Multiplicative inverse of z_3 *is* $\frac{1}{z_3} = \frac{1}{1+i}$

$$= \frac{1}{1+i} \times \frac{1-i}{1-i} = \frac{1-i}{(1+i)(1-i)}$$

$$= \frac{1-i}{1^2+1^2} = \frac{1-i}{2}$$

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EXERCISE 2.4

Example 2.3: Write $\frac{3+4i}{5-12i}$ in the $x+iy$ form hence find its real and imaginary part

$(a+ib)(a-ib) = a^2+b^2$

$$\frac{3+4i}{(5-12i)} \times \frac{(5+12i)}{(5+12i)} = \frac{(3+4i)(5+12i)}{(5-12i)(5+12i)}$$

$$= \frac{15+36i+20i+48i^2}{5^2+12^2}$$

$$= \frac{15+36i+20i+48(-1)}{25+144}$$

$$= \frac{15+56i-48}{169} = \frac{56i-33}{169} = \frac{56i}{169} - \frac{33}{169}$$

$$x+iy = \frac{-33}{169} + i\frac{56}{169}$$

Example 2.4: Simplify: $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3$

$(a+ib)(a-ib) = a^2+b^2$

Take: $\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{(1+i)^2}{1^2+1^2}$

$$= \frac{1^2+2(1)(i)+i^2}{1+1} = \frac{1+2i-1}{2} = \frac{2i}{2} = i$$

$$\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{(1-i)^2}{1^2+1^2}$$

$$= \frac{1^2-2(1)(i)+i^2}{1+1} = \frac{1-2i-1}{2} = \frac{-2i}{2} = -i$$

$$\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = i^3-(-i)^3 = -i-(-i)^3 = -i-(-(-i))$$

$$= -i-i = -2i$$

Example 2.5: If $\frac{z+3}{z-5i} = \frac{1+4i}{2}$, find the complex number z

$$\frac{z+3}{z-5i} = \frac{1+4i}{2} \Longrightarrow 2(z+3) = (1+4i)(z-5i)$$

$$2z+6 = z-5i+4iz-20i^2 \Longrightarrow 2z+6 = z-5i+4iz-20(-1)$$

$$2z+6 = z-5i+4iz+20 \Longrightarrow 2z-z = -5i+4iz+20-6$$

$$z = -5i+4iz+14 \Longrightarrow z-4iz = -5i+14$$

$$z(1-4i) = -5i+14 \Longrightarrow z = \frac{-5i+4}{1-4i}$$

$$z = \frac{(-5i+14)(1+4i)}{(1-4i)(1+4i)} = \frac{-5i-20i^2+14+56i}{1^2+4^2}$$

$$= \frac{20+14+51i}{1+16} = \frac{34+51i}{17}$$

$$z = \frac{34}{17} + \frac{51i}{17} \Longrightarrow z = 2+3i$$

***Example 2.6*: *If* $z_1 = 3-2i$ *and* $z_2 = 6+4i$ *find* $\frac{z_1}{z_2}$**

Given: $z_1 = 3-2i, z_2 = 6+4i$

$$\frac{z_1}{z_2} = \frac{3-2i}{6+4i} = \frac{(3-2i)(6-4i)}{(6+4i)(6-4i)} = \frac{18-12i-12i+8i^2}{6^2+4^2}$$

$$= \frac{18-24i+8(-1)}{36+16} = \frac{18-24i-8}{52}$$

$$= \frac{10-24i}{52} = \frac{10}{52} - \frac{24}{52}i = \frac{5}{26} - \frac{6}{13}i$$

***Example 2.7*: *Find* z^{-1}, *if* $z = (2+3i)(1-i)$**

$$Z = (2+3i)(1-i)$$

$$= 2-2i+3i-3i^2 = 2+3i-2i-3(-1)$$

$$= 2+i+3$$

$$z = 5+i$$

$$z^{-1} = \frac{1}{z} = \frac{1}{5+i}$$

$$= \frac{1(5-i)}{(5+i)(5-i)} = \frac{5-i}{5^2+1^2} = \frac{5-i}{25+1}$$

$$= \frac{5-i}{26} = \frac{5}{26} - i\frac{1}{26}$$

Example 2.8* : *Show that i*) $(2+i\sqrt{3})^{10} + (2-i\sqrt{3})^{10}$ *is real

ii*) $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+i}\right)^{15}$ *is purely imaginary.

z is purely real if and only if $z = \bar{z}$

$$\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$$

***i*) $(2+i\sqrt{3})^{10} + (2-i\sqrt{3})^{10}$**

Let $z = (2+i\sqrt{3})^{10} + (2-i\sqrt{3})^{10}$

$$\bar{z} = \overline{(2+i\sqrt{3})^{10} + (2-i\sqrt{3})^{10}} = \overline{(2+i\sqrt{3})^{10}} + \overline{(2-i\sqrt{3})^{10}}$$

$$\bar{z} = (2-i\sqrt{3})^{10} + (2+i\sqrt{3})^{10}$$

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$\overline{z} = \left(2+i\sqrt{3}\right)^{10} + \left(2-i\sqrt{3}\right)^{10}$

$\overline{z} = z$ *is a real*

ii) $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$

z is purely imaginary if and only if $z = -\overline{z}$

$$Let\ z = \left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+i}\right)^{15}$$

$$\frac{19+9i}{5-3i} = \frac{(19+9i)\,(5+3i)}{(5-3i)\,(5+3i)} = \frac{95+57i+45i+27i^2}{5^2+3^2}$$

$$= \frac{95+102i+27(-1)}{25+9} = \frac{95+102i-27}{34}$$

$$= \frac{68+102i}{34} = \frac{68}{34} + \frac{102i}{34} = 2+3i$$

$\frac{19+9i}{5-3i} = 2+3i$

$$\frac{8+i}{1+2i} = \frac{(8+i)(1-2i)}{(1+2i)(1-2i)} = \frac{8-16i+i-2i^2}{1^2+2^2}$$

$$= \frac{8-16i+i-2(-1)}{1+4} = \frac{8-15i+2}{5} = \frac{10-15i}{5} = \frac{10}{5} - \frac{15i}{5}$$

$\frac{8+i}{1+2i} = 2-3i$

$$z = \left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+i}\right)^{15}$$

$z = (2+3i)^{15} - (2-3i)^{15}$

$\overline{z} = \overline{(2+3i)^{15} - (2-3i)^{15}} = \overline{(2+3i)^{15}} - \overline{(2-3i)^{15}}$

$\overline{z} = (2-3i)^{15} - (2+3i)^{15} \Rightarrow -\overline{z} = -(2-3i)^{15} + (2+3i)^{15}$

$-\overline{z} = (2+3i)^{15} - (2-3i)^{15}$

$-\overline{z} = z$ *is purely imaginary.*

1. *Write the following in the rectangular form*:

i) $\overline{(5+9i)+(2-4i)}$

$\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$

$= \overline{(5+9i)+(2-4i)} = \overline{(5+9i)} + \overline{(2-4i)}$

$= 5-9i+2+4i = 7-5i$

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$\boldsymbol{ii)\ \dfrac{10-5i}{6+2i}}$

$$\frac{10-5i}{6+2i}=\frac{(10-5i)(6-2i)}{(6+2i)(6-2i)}=\frac{60-20i-30i+10i^2}{6^2+2^2}$$

$$=\frac{60-50i-10}{36+4}=\frac{50-50i}{40}$$

$$=\frac{50(1-i)}{40}=\frac{5}{4}(1-i)=\frac{5}{4}-\frac{5}{4}i$$

$\boldsymbol{iii)\ \overline{3i}+\dfrac{1}{2-i}}$

$$\overline{3i}+\frac{1}{2-i}=-3i+\frac{1}{2-i}$$

$$=-3i+\frac{2+i}{(2-i)(2+i)}$$

$$=-3i+\frac{2+i}{2^2+1^2}=-3i+\frac{2+i}{5}$$

$$=\frac{-15i+2+i}{5}=\frac{-14i+2}{5}=\frac{2}{5}-\frac{14}{5}i$$

$\boldsymbol{2. If\ z=x+iy, find\ the\ following\ in\ rectangular\ form.}$

$\boldsymbol{i)\ Re\left(\dfrac{1}{z}\right)}$

$z=x+iy$

$Re(z)=x, Im(z)=y$

$$\frac{1}{z}=\frac{1}{x+iy}$$

$$=\frac{x-iy}{(x+iy)(x-iy)}=\frac{x-iy}{x^2+y^2}$$

$$\frac{1}{z}=\frac{x}{x^2+y^2}-\frac{iy}{x^2+y^2}$$

$$Re\left(\frac{1}{z}\right)=\frac{x}{x^2+y^2}$$

$\boldsymbol{ii)\ Re(\overline{iz})}$

$z=x+iy$

$iz=i(x+iy)=ix+\underset{}{\overset{(-1)}{i^2}}y$

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$iz = ix - y \Longrightarrow iz = -y + ix$

$Re(iz) = -y$

iii) $\boldsymbol{Im(3z + 4\bar{z} - 4i)}$

$z = x + iy$

$3z + 4\bar{z} - 4i = 3(x + iy) + 4(\overline{x + iy}) - 4i$

$= 3(x + iy) + 4\,(x - iy) - 4i$

$= 3x + i3y + 4x - i4y - 4i$

$= 7x - iy - 4i = 7x - i(y + 4)$

$3z + 4\bar{z} - 4i = 7x - i(y + 4)$

$Im(3z + 4\bar{z} - 4i) = -(y + 4)$

3. ***If*** $\boldsymbol{z_1 = 2 - i, z_2 = -4 + 3i}$, ***find the inverse of*** $\boldsymbol{z_1 z_2}$ ***and*** $\boldsymbol{\frac{z_1}{z_2}}$

Given: $z_1 = 2 - i, z_2 = -4 + 3i$

$z_1 z_2 = (2 - i)(-4 + 3i)$

$= -8 + 6i + 4i - 3i^2$

$= -8 + 10i + 3 = -5 + 10i$

$z_1 z_2 = -(5 - 10i)$

Inverse of $z_1 z_2$

$$\frac{1}{z_1 z_2} = \frac{1}{-(5 - 10i)} = \frac{-1}{(5 - 10i)} = \frac{-1(5 + 10i)}{(5 - 10i)(5 + 10i)}$$

$$= \frac{-5 - 10i}{5^2 + 10^2} = \frac{-5 - 10i}{25 + 100} = \frac{-5 - 10i}{125}$$

$$= \frac{-5}{125} - \frac{10i}{125} = \frac{-1}{25} - \frac{2}{25}i$$

Inverse of $\frac{z_1}{z_2}$

$$\left(\frac{z_1}{z_2}\right)^{-1} = \frac{z_2}{z_1} = \frac{-4 + 3i}{2 - i} = \frac{(-4 + 3i)(2 + i)}{(2 - i)(2 + i)}$$

$$= \frac{-8 - 4i + 6i + 3i^2}{2^2 + 1^2} = \frac{-8 + 2i - 3}{4 + 1}$$

$$= \frac{-11 + 2i}{5} = \frac{-11}{5} + \frac{2}{5}i$$

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4. The complex numbers u, v and w are related by $\frac{1}{u}=\frac{1}{v}+\frac{1}{w}$, if $v=3-4i$ and $w=4+3i$. find u in rectangular form

$$\frac{1}{u}=\frac{1}{v}+\frac{1}{w}\Longrightarrow \frac{1}{u}=\frac{w+v}{wv}\Longrightarrow u=\frac{vw}{w+v}$$

$$u=\frac{(3-4i)(4+3i)}{(4+3i)+(3-4i)}=\frac{12+9i-16i-12i^2}{7-i}$$

$$=\frac{12-7i+12}{7-i}=\frac{24-7i}{7-i}$$

$$u=\frac{(24-7i)(7+i)}{(7-i)(7+i)}=\frac{168+24i-49i-7i^2}{7^2+1^2}=\frac{168-25i+7}{49+1}$$

$$=\frac{175-25i}{50}=\frac{175}{50}-\frac{25}{50}i$$

$$u=\frac{7}{2}-\frac{1}{2}i$$

5. Prove the following properties:

i) z is real if and only if $z=\bar{z}$

Let $z=x+iy, \bar{z}=x-iy$

$z=\bar{z}$

$x+iy=x-iy$

$x+iy-x+iy=0$

$i2y=0\Longrightarrow \boxed{y=0}$

$z=x+iy \Longrightarrow z=x+0$

$z=x$

z is real

ii) $Re(z)=\frac{z+\bar{z}}{2}$ and $Im(z)=\frac{z-\bar{z}}{2i}$

Let $z=x+iy, \bar{z}=x-iy$

$z+\bar{z}= x+iy+x-iy$

$z+\bar{z}=2x \Longrightarrow \frac{z+\bar{z}}{2}=\frac{2x}{2}=x$

$\frac{z+\bar{z}}{2}=Re(z)$

$Im(z)=\frac{z-\bar{z}}{2i}$

Let $z=x+iy, \bar{z}=x-iy$

$$z - \overline{z} = x + iy - x + iy$$

$$z - \overline{z} = 2iy \implies \frac{z-\overline{z}}{2i} = \frac{2iy}{2i} = y$$

$$\frac{z-\overline{z}}{2i} = Im(z)$$

6. *Find the least value of two positive integer n for which $\left(\sqrt{3}+i\right)^n$ i) Real ii) Purely imaginary.*

$$\left(\sqrt{3}+i\right)^2 = \left(\sqrt{3}\right)^2 + 2\left(\sqrt{3}\right)i + i^2 = 3 + 2\sqrt{3}i - 1$$

$$\left(\sqrt{3}+i\right)^2 = 2 + 2\sqrt{3}i \implies \left(\sqrt{3}+i\right)^2 = 2\left(1+\sqrt{3}i\right)$$

$$\left(\sqrt{3}+i\right)^3 = \left(\sqrt{3}+i\right)^2\left(\sqrt{3}+i\right) \implies \left(\sqrt{3}+i\right)^3 = 2\left(1+\sqrt{3}i\right)\left(\sqrt{3}+i\right)$$

$$\left(\sqrt{3}+i\right)^3 = 2\left(\sqrt{3}+i+3i+\sqrt{3}i^2\right) = 2\left(\sqrt{3}+i+3i-\sqrt{3}\right)$$

$$= 2(i+3i)$$

$$\left(\sqrt{3}+i\right)^3 = 2(4i)$$

$\left(\sqrt{3}+i\right)^3 = 8i$ *is purely imaginary* $\quad \therefore$ *Least value of n is* 3.

$$\left(\sqrt{3}+i\right)^3\left(\sqrt{3}+i\right)^3 = (8i)(8i)$$

$\left(\sqrt{3}+i\right)^6 = 64i^2 = 64(-1) = -64$ *is real*

$\therefore$ *The least value of n is* 6.

7. *Show that i)* $\left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}$ *is purely imaginary.*

Let $z = \left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}$

> $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

> $z = -\overline{z}$

$$\overline{z} = \overline{\left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}}$$

$$= \overline{\left(2+i\sqrt{3}\right)^{10}} - \overline{\left(2-i\sqrt{3}\right)^{10}}$$

$$\overline{z} = \left(2-i\sqrt{3}\right)^{10} - \left(2+i\sqrt{3}\right)^{10} \implies -\overline{z} = -\left(2-i\sqrt{3}\right)^{10} + \left(2+i\sqrt{3}\right)^{10}$$

$-\overline{z} = \left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10} \implies -\overline{z} = z$ ***is purely imaginary***

ii) $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ ***is real.***

Let $z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$

> *z is purely real if and only if* $z = \overline{z}$

Take: $\frac{19-7i}{9+i} = \frac{(19-7i)(9-i)}{(9+i)(9-i)} = \frac{171 - 19i - 63i + 7i^2}{9^2+1^2}$ $\quad (i^2 = -1)$

$$= \frac{171 - 82i - 7}{81 + 1} = \frac{164 - 82i}{82} = \frac{164}{82} - \frac{82}{82} i = 2 - i$$

$$\frac{19 - 7i}{9 + i} = 2 - i$$

$$Take: \frac{20 - 5i}{7 - 6i} = \frac{(20 - 5i)(7 + 6i)}{(7 - 6i)(7 + 6i)} = \frac{140 + 120i - 35i - 30i^2}{7^2 + 6^2}$$

$$= \frac{140 + 120i - 35i + 30}{49 + 36} = \frac{170 + 85i}{85} = \frac{170}{85} + \frac{85}{85} i = 2 + i$$

$$\boxed{\frac{20 - 5i}{7 - 6i} = 2 + i}$$

$$z = \left(\frac{19 - 7i}{9 + i}\right)^{12} + \left(\frac{20 - 5i}{7 - 6i}\right)^{12}$$

$$z = (2 - i)^{12} + (2 + i)^{12}$$

$$\bar{z} = \overline{(2 - i)^{12} + (2 + i)^{12}} = \overline{(2 - i)^{12}} + \overline{(2 + i)^{12}}$$

$$\bar{z} = (2 + i)^{12} + (2 - i)^{12} \Longrightarrow \bar{z} = (2 - i)^{12} + (2 + i)^{12} = z$$

$$\bar{z} = z \text{ is a real}$$

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Exercise 2.5

Example* 2.9**: ***If $z_1 = 3 + 4i, z_2 = 5 - 12i$ ***and*** $z_3 = 6 + 8i$ ***find*** $|z_1|, |z_2|, |z_3|, |z_1 + z_2|, |z_2 - z_3|, |z_1 + z_3|$

Given: $z_1 = 3 + 4i, z_2 = 5 - 12i, z_3 = 6 + 8i$

$$|z_1| = |3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25}$$

$$\boxed{|z_1| = 5}$$

$$|z_2| = |5 - 12i| = \sqrt{5^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169}$$

$$\boxed{|z_2| = 13}$$

$$|z_3| = |6 + 8i| = \sqrt{6^2 + (8)^2} = \sqrt{36 + 64} = \sqrt{100}$$

$$\boxed{|z_3| = 10}$$

$$|z_1 + z_2| = |3 + 4i + 5 - 12i|$$

$$= |8 - 8i| = \sqrt{8^2 + (-8)^2} = \sqrt{64 + 64} = \sqrt{128}$$

$$= \sqrt{64 \times 2} = 8\sqrt{2}$$

$$|z_1 + z_2| = 8\sqrt{2}$$

$$|z_2 - z_3| = |5 - 12i - (6 + 8i)|$$

$$= |5 - 12i - 6 - 8i| = |-1 - 20i|$$

$$= \sqrt{(-1)^2 + (-20)^2} = \sqrt{1 + 400} = \sqrt{401}$$

$$|z_2 - z_3| = \sqrt{401}$$

$$|z_1 - z_3| = |3 + 4i + 6 + 8i| = |9 + 12i|$$

$$= \sqrt{9^2 + 12^2} = \sqrt{81 + 144} = \sqrt{225}$$

$$|z_1 - z_3| = 15$$

Example* 2.10**: ***Find the following i) $\left|\dfrac{2 + i}{-1 + 2i}\right|$

ii) $|\overline{(1 + i)}(2 + 3i)(4i - 3)|$ ***iii***) $\left|\dfrac{i(2 + i)^3}{(1 + i)^2}\right|$

i) $\left|\dfrac{2 + i}{-1 + 2i}\right|$

$$\boxed{\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}}$$

$$\left|\frac{2 + i}{-1 + 2i}\right| = \frac{|2 + i|}{|-1 + 2i|} = \frac{\sqrt{2^2 + 1^2}}{\sqrt{(-1)^2 + (2)^2}}$$

$$= \frac{\sqrt{4 + 1}}{\sqrt{1 + 4}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

$\boldsymbol{ii})\ |(\overline{\mathbf{1}+\boldsymbol{i}})(\mathbf{2}+\mathbf{3}\boldsymbol{i})(\mathbf{4}\boldsymbol{i}-\mathbf{3})|$

$$\boxed{|z_1z_2| = |z_1||z_2|}$$

$$\boxed{|z| = |\bar{z}|}$$

$= |(\overline{1+i})(2+3i)(4i-3)|$

$= |(1-i)|\,|2+3i||4i-3| = |\overline{1+i}||2+3i||4i-3|$

$= |1+i||2+3i||4i-3| = \left(\sqrt{1^2+1^2}\right)\left(\sqrt{2^2+3^2}\right)\left(\sqrt{4^2+(-3)^2}\right)$

$= \left(\sqrt{1+1}\right)\left(\sqrt{4+9}\right)\left(\sqrt{16+9}\right) = \sqrt{2}\times\sqrt{13}\times\sqrt{25}$

$= \sqrt{2\times 13}\times 5 = 5\sqrt{26}$

$\boldsymbol{iii})\ \left|\dfrac{\boldsymbol{i}(\mathbf{2}+\boldsymbol{i})^{\mathbf{3}}}{(\mathbf{1}+\boldsymbol{i})^{\mathbf{2}}}\right|$

$$\boxed{|z^n| = |z|^n}$$

$$\boxed{\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}}$$

$\left|\dfrac{i(2+i)^3}{(1+i)^2}\right| = \dfrac{|i(2+i)^3|}{|(1+i)^2|}$

$= \dfrac{|i||2+i|^3}{|(1+i)|^2} = \dfrac{\sqrt{1^2}\left(\sqrt{2^2+1^2}\right)^3}{\left(\sqrt{1^2+1^2}\right)^2}$

$= \dfrac{1\left(\sqrt{4+1}\right)^3}{\left(\sqrt{1+1}\right)^2} = \dfrac{\left(\sqrt{5}\right)^3}{\left(\sqrt{2}\right)^2} = \dfrac{\sqrt{5}\times\sqrt{5}\times\sqrt{5}}{\sqrt{2}\times\sqrt{2}} = \dfrac{5\sqrt{5}}{2}$

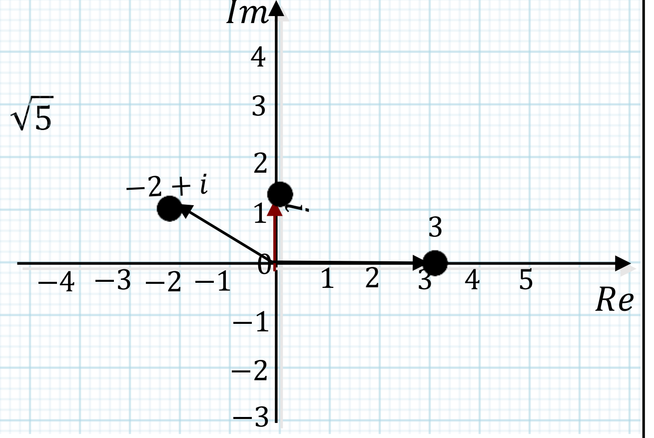
***Example 2.11**: Which one of the point is $i, -2+i$ and 3 is farthest from the origin.*

$|i| = \sqrt{1^2} = 1$

$|-2+i| = \sqrt{(-2)^2+1^2} = \sqrt{4+1} = \sqrt{5}$

$|3| = \sqrt{3^2} = 3$

The farthest point from orgin is 3.



***Example 2.12**: If z_1, z_2 and z_3 are complex numbers such that*

$$|\boldsymbol{z_1}| = |\boldsymbol{z_2}| = |\boldsymbol{z_3}| = |\boldsymbol{z_1}+\boldsymbol{z_2}+\boldsymbol{z_3}| = \mathbf{1}, \textit{find the value of } \left|\frac{\mathbf{1}}{\boldsymbol{z_1}}+\frac{\mathbf{1}}{\boldsymbol{z_2}}+\frac{\mathbf{1}}{\boldsymbol{z_3}}\right|$$

$\lvert z_1\rvert = 1 \Rightarrow \lvert z_1\rvert^2 = 1$ $z_1\overline{z_1} = 1 \Rightarrow \overline{z_1} = \frac{1}{z_1}$	$\lvert z_2\rvert = 1 \Rightarrow \lvert z_2\rvert^2 = 1$ $z_2\overline{z_2} = 1 \Rightarrow \overline{z_2} = \frac{1}{z_2}$	$\lvert z_3\rvert = 1 \Rightarrow \lvert z_3\rvert^2 = 1$ $z_3\overline{z_3} = 1 \Rightarrow \overline{z_3} = \frac{1}{z_3}$

$\left|\dfrac{1}{z_1}+\dfrac{1}{z_2}+\dfrac{1}{z_3}\right| = |\overline{z_1}+\overline{z_2}+\overline{z_3}| = |\overline{z_1+z_2+z_3}|$

$$\boxed{\overline{z_1}+\overline{z_2} = \overline{z_1+z_2}}$$

$$\boxed{|z| = |\bar{z}|}$$

$$\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right| = |z_1+z_2+z_3| = 1$$

$|z|^2 = z\bar{z}$

Example 2.13: If $|z| = 2$, show that $3 \le |z+3+4i| \le 7$

$$|\underbrace{z}_{z_1} + \underbrace{3+4i}_{z_2}| \le |z| + |3+4i|$$

$$\le 2 + \sqrt{3^2+4^2}$$

$$\le 2 + \sqrt{25}$$

$$\le 2 + 5$$

$$|z+3+4i| \le 7 \ \dots(1)$$

$|z_1+z_2| \le |z_1| + |z_2|$

$|x+iy| = \sqrt{x^2+y^2}$

$|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$

$$|z+3+4i| \ge \big||z| - |3+4i|\big|$$

$$\ge \left|2 - \sqrt{3^2+4^2}\right|$$

$$\ge |2 - \sqrt{9+16}|$$

$$\ge |2 - \sqrt{25}|$$

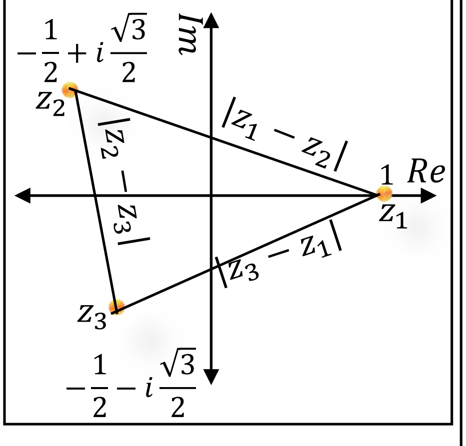
$$\ge |2-5|$$

$$|z+3+4i| \ge |-3|$$

$$|z+3+4i| \ge |3| \ \dots(2)$$

From (1) *and* (2) $3 \le |z+3+4i| \le 7$

Example 2.14: Show that the points $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$ are the vertices of an equilateral triangle



Let $z_1 = 1,\ z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ z_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$$|z_1 - z_2| = \left|1 - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right| = \left|1 + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right|$$

$$= \left|\frac{3}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4}+\frac{3}{4}} = \sqrt{\frac{12}{4}}$$

$$|z_1 - z_2| = \sqrt{3}$$

$$|z_2 - z_3| = \left|-\frac{1}{2} + i\frac{\sqrt{3}}{2} - \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right|$$

$$= \left|-\frac{1}{2} + i\frac{\sqrt{3}}{2} + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right| = \left|i\frac{\sqrt{3}}{2} + i\frac{\sqrt{3}}{2}\right| = \left|\frac{2i\sqrt{3}}{2}\right|$$

$$= |i\sqrt{3}| = \sqrt{(\sqrt{3})^2}$$

$$|z_2 - z_3| = \sqrt{3}$$

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$$|z_3 - z_1| = \left|-\frac{1}{2} - i\frac{\sqrt{3}}{2} - 1\right| = \left|-\frac{3}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}}$$

$$|z_3 - z_1| = \sqrt{3}$$

$$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$$

$\therefore$ *Given points form a equilateral triangle.*

Example 2.15: **Let z_1, z_2 and z_3 be complex number such that $|z_1| = |z_2| = |z_3| = r > 0$ and $z_1 + z_2 + z_3 \neq 0$.**

prove that $\left|\frac{z_1 z_2 + z_2 z_3 + z_3 z_1}{z_1 + z_2 + z_3}\right|$

$$|z|^2 = z\bar{z}$$

$$\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}$$

$$|z| = |\bar{z}|$$

$$|z_1| = |z_2| = |z_3| = r$$

$$|z_1| = r \Longrightarrow |z_1|^2 = r^2$$

$$z_1\overline{z_1} = r^2 \Longrightarrow z_1 = \frac{r^2}{\overline{z_1}}$$

Similarly

$$z_2 = \frac{r^2}{\overline{z_2}}, \quad z_3 = \frac{r^2}{\overline{z_3}}$$

$$z_1 + z_2 + z_3 = \frac{r^2}{\overline{z_1}} + \frac{r^2}{\overline{z_2}} + \frac{r^2}{\overline{z_3}}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$z_1 + z_2 + z_3 = r^2\left(\frac{1}{\overline{z_1}} + \frac{1}{\overline{z_2}} + \frac{1}{\overline{z_3}}\right)$$

$$z_1 + z_2 + z_3 = r^2\left(\frac{\overline{z_2}\,\overline{z_3} + \overline{z_1}\,\overline{z_3} + \overline{z_1}\,\overline{z_2}}{\overline{z_1}\,\overline{z_2}\,\overline{z_3}}\right)$$

$$z_1 + z_2 + z_3 = r^2\left(\frac{\overline{z_2 z_3 + z_1 z_3 + z_1 z_2}}{\overline{z_1 z_2 z_3}}\right)$$

$$|z_1 + z_2 + z_3| = |r^2|\left|\frac{\overline{z_2 z_3 + z_1 z_3 + z_1 z_2}}{\overline{z_1 z_2 z_3}}\right| = r^2\frac{|\overline{z_2 z_3 + z_1 z_3 + z_1 z_2}|}{|\overline{z_1 z_2 z_3}|}$$

$$|z_1 + z_2 + z_3| = r^2\frac{|z_2 z_3 + z_1 z_3 + z_1 z_2|}{|z_1 z_2 z_3|}$$

$$|\bar{z}| = |z|$$

$$|z_1 + z_2 + z_3| = r^2\frac{|z_2 z_3 + z_1 z_3 + z_1 z_2|}{|z_1||z_2||z_3|}$$

$$|z_1+z_2+z_3| = r^2\frac{|z_2z_3+z_1z_3+z_1z_2|}{r\times r\times r} = \frac{|z_2z_3+z_1z_3+z_1z_2|}{r}$$

$$|z_1+z_2+z_3| = \frac{|z_1z_2+z_2z_3+z_1z_3|}{r}$$

$$r = \frac{|z_1z_2+z_2z_3+z_1z_3|}{|z_1+z_2+z_3|}$$

Example 2.16: Show that the equation $z^2 = \bar{z}$ has four solutions

$$z^2 = \bar{z}$$

$$|z^2| = |\bar{z}|$$

$$|z|^2 = |z|$$

$\lvert z^n\rvert = \lvert z\rvert^n$
$\lvert\bar{z}\rvert = \lvert z\rvert$

$$|z|^2 - |z| = 0 \implies |z|(|z|-1) = 0$$

$$|z| = 0; \quad |z|-1 = 0$$

$$z = 0; \quad |z| = 1$$

$$|z|^2 = 1 \implies z\bar{z} = 1$$

$$\bar{z} = \frac{1}{z} \implies z^2 = \frac{1}{z} \implies z^3 = 1$$

It has 3 non − zero solution, hence including zero solution, there are four solutions.

1. *Find the modulus of the following complex numbers*

(*i*) $\frac{2i}{3+4i}$

$\left\lvert\frac{Z_1}{Z_2}\right\rvert = \frac{\lvert Z_1\rvert}{\lvert Z_2\rvert}$

$$\left|\frac{2i}{3+4i}\right| = \frac{|2i|}{|3+4i|} = \frac{\sqrt{0^2+2^2}}{\sqrt{3^2+4^2}} = \frac{\sqrt{4}}{\sqrt{9+16}}$$

$$= \frac{2}{\sqrt{25}} = \frac{2}{5}$$

(*ii*) $\frac{2-i}{1+i} + \frac{1-2i}{1-i}$

$$= \frac{2-i}{1+i} + \frac{1-2i}{1-i} = \frac{(2-i)(1-i)+(1-2i)(1+i)}{(1+i)(1-i)}$$

$$= \frac{2-2i-i+\overset{(-1)}{i^2}+1+i-2i-2\overset{(-1)}{i^2}}{1^2+1^2} = \frac{2-3i-1+1-i+2}{1+1}$$

$$= \frac{4-4i}{2} = \frac{4}{2} - \frac{4}{2}i = 2-2i = |2-2i|$$

$$= \sqrt{2^2+(-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

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(iii) $(1-i)^{10}$

$$|(1-i)^{10}| = |1-i|^{10}$$
$$= \left(\sqrt{1^2+(-1)^2}\right)^{10} = \left(\sqrt{1+1}\right)^{10}$$

$\boxed{|Z^n| = |Z|^n}$

$$= \left(\sqrt{2}\right)^{10} = \left(\sqrt{2}\right)^{2\times5} = 2^5 = 32$$

(iv) $2i(3-4i)(4-3i)$

$$= |2i(3-4i)(4-3i)| = |2i||3-4i||4-3i|$$
$$= \left(\sqrt{0^2+2^2}\right)\left(\sqrt{3^2+(-4)^2}\right)\left(\sqrt{4^2+(-3)^2}\right)$$
$$= (\sqrt{4})(\sqrt{9+16})(\sqrt{16+9}) = 2(\sqrt{25})(\sqrt{25})$$
$$= 2(5)(5) = 50$$

2. *For any two complex numbers z_1 and z_2, such that $|z_1| = |z_2| = 1$ and $z_1z_2 \neq -1$, then show that $\frac{z_1+z_2}{1+z_1z_2}$ is a real number*

Given $|z_1| = 1$ *and* $|z_2| = 1$

$\boxed{|z|^2 = z\bar{z}}$

To prove : $\frac{z_1+z_2}{1+z_1z_2}$ *is a real number*

$$|z_1| = 1$$
$$|z_1|^2 = 1 \Rightarrow z_1\overline{z_1} = 1 \Rightarrow z_1 = \frac{1}{\overline{z_1}}$$

Similarly $z_2 = \frac{1}{\overline{z_2}}$

$$\frac{z_1+z_2}{1+z_1z_2} = \frac{\frac{1}{\overline{z_1}}+\frac{1}{\overline{z_2}}}{1+\frac{1}{\overline{z_1}}\times\frac{1}{\overline{z_2}}} = \frac{\frac{\overline{z_1}+\overline{z_2}}{\overline{z_1}\,\overline{z_2}}}{\frac{1}{1}+\frac{1}{\overline{z_1}\,\overline{z_2}}}$$

$$= \frac{\frac{\overline{z_1}+\overline{z_2}}{\overline{z_1}\,\overline{z_2}}}{\frac{\overline{z_1}\,\overline{z_2}+1}{\overline{z_1}\,\overline{z_2}}} = \frac{\overline{z_1}+\overline{z_2}}{1+\overline{z_1}\,\overline{z_2}}$$

$\boxed{|z|^2 = z\bar{z}}$

$\boxed{z+\bar{z} = 2Re(z)}$

$$= \frac{(\overline{z_1}+\overline{z_2})(1+z_1z_2)}{(1+\overline{z_1}\,\overline{z_2})(1+z_1z_2)} = \frac{\overline{z_1}+\overline{z_1}z_1z_2+\overline{z_2}+z_1z_2\overline{z_2}}{1+z_1z_2+\overline{z_1}\,\overline{z_2}+\overline{z_1}z_1z_2\overline{z_2}}$$

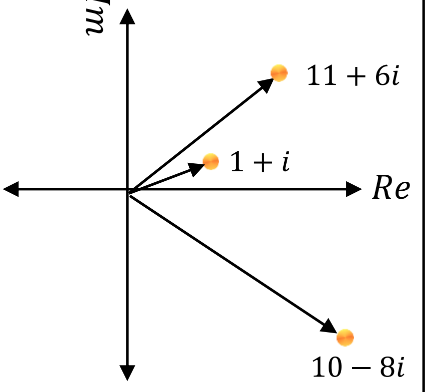
$$= \frac{\overline{z_1}+|z_1|^2z_2+\overline{z_2}+z_1|z_2|^2}{1+z_1z_2+\overline{z_1}\,\overline{z_2}+|z_1|^2|z_2|^2}$$ *where* $|z_1| = 1$ *and* $|z_2| = 1$

$$= \frac{\overline{z_1}+(1)^2z_2+\overline{z_2}+z_1(1)^2}{1+z_1z_2+\overline{z_1}\,\overline{z_2}+(1)^2(1)^2} = \frac{\overline{z_1}+z_2+\overline{z_2}+z_1}{1+z_1z_2+\overline{z_1}\,\overline{z_2}+1} = \frac{z_1+\overline{z_1}+z_2+\overline{z_2}}{2+z_1z_2+\overline{z_1z_2}}$$

$$= \frac{2Re(z_1) + 2Re(z_2)}{2 + 2Re(z_1 z_2)} \text{ is a real number.}$$

3. Which one of the points $10 - 8i, 11 + 6i$ is closest to $1 + i$.

Given points : $10 - 8i, 11 + 6i$



$$|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{1 + 1} = \sqrt{2}$$

$$|10 - 8i| = \sqrt{10^2 + (-8)^2} = \sqrt{100 + 64} = \sqrt{164}$$

$$|11 - 6i| = \sqrt{11^2 + (6)^2} = \sqrt{121 + 36} = \sqrt{157}$$

$\therefore$ $11 + 6i$ *is closest to* $1 + i$.

4. If $|z| = 3$, show that $7 \leq |z + 6 - 8i| \leq 13$.

$$\boxed{|z_1 + z_2| \leq |z_1| + |z_2|}$$

Given: $|z| = 3$

$$|z + 6 - 8i| \leq |z| + |6 - 8i|$$

$$\leq 3 + \sqrt{6^2 + (-8)^2} \implies |z + 6 - 8i| \leq 3 + \sqrt{36 + 64}$$

$$|z + 6 - 8i| \leq 3 + \sqrt{100} \implies |z + 6 - 8i| \leq 3 + 10$$

$$|z + 6 - 8i| \leq 13 \quad \dots(1)$$

$$|z + 6 - 8i| \geq \big||z| - |6 - 8i|\big|$$

$$\geq \left|3 - \sqrt{6^2 + (-8)^2}\right| \implies |z + 6 - 8i| \geq \left|3 - \sqrt{36 + 64}\right|$$

$$|z + 6 - 8i| \geq \left|3 - \sqrt{100}\right| \implies |z + 6 - 8i| \geq |3 - 10|$$

$$|z + 6 - 8i| \geq |7| \implies |z + 6 - 8i| \geq 7 \quad \dots(2)$$

From (1) *and* (2) $7 \leq |z + 6 - 8i| \leq 13$

5. If $|z| = 1$, show that $2 \leq |z^2 - 3| \leq 4$.

$$\boxed{|z_1 + z_2| \leq |z_1| + |z_2|}$$

$$\boxed{|z^n| = |z|^n}$$

$$\boxed{|z_1 - z_2| \geq \big||z_1| - |z_2|\big|}$$

Given: $|z| = 1, |z^2 - 3|$

$$|z^2 + (-3)| \leq |z^2| + |-3|$$

$$\leq |z|^2 + 3$$

$$\leq (1)^2 + 3 \implies |z^2 + (-3)| \leq 1 + 3$$

$$|z^2 - 3| \leq 4 \dots(1)$$

$$|z^2 - 3| \geq \big||z^2| - |-3|\big| \implies |z^2 - 3| \geq |(1)^2 - 3|$$

$$|z^2 - 3| \geq |1 - 3| \implies |z^2 - 3| \geq |-2|$$

$$|z^2 - 3| \geq 2 \dots(2)$$

From (1) *and* (2) $2 \leq |z^2 - 3| \leq 4$

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6. If $\left|z-\frac{2}{z}\right| = 2$, show that the greatest and least value of $|z|$ is $\sqrt{3}+1$ and $\sqrt{3}-1$ respectively.

$$|z| = \left|z-\frac{2}{z}+\frac{2}{z}\right| \Rightarrow |z| \le \left|z-\frac{2}{z}\right| + \left|\frac{2}{z}\right|$$

$|z_1+z_2| \le |z_1|+|z_2|$

$$where \left|z-\frac{2}{z}\right| = 2$$

$$|z| \le 2+\frac{2}{|z|} \Rightarrow |z| \le \frac{2|z|+2}{|z|}$$

$$|z|^2 \le 2|z|+2 \Rightarrow |z|^2 - 2|z| \le 2$$

$$|z|^2 - 2|z| + 1^2 \le 2+1^2 \Rightarrow (|z|-1)^2 \le 3$$

$$|z|-1 \le \pm\sqrt{3} \Rightarrow |z|-1 \le \sqrt{3}, |z|-1 \le -\sqrt{3}$$

$$|z| \le 1-\sqrt{3}, |z| \le 1+\sqrt{3}$$

$$1-\sqrt{3} \le |z| \le 1+\sqrt{3}$$

7. If z_1, z_2 and z_3 are three complex numbers such that $|z_1| = 1, |z_2| = 2, |z_3| = 3$ and $|z_1+z_2+z_3| = 1$, show that $|9z_1z_2+4z_1z_3+z_2z_3| = 6$

Given:

$$|z_1| = 1,\ |z_1|^2 = 1 \Rightarrow z_1\overline{z_1} = 1 \Rightarrow \overline{z_1} = \frac{1}{z_1}$$

$$|z_2| = 2 \Rightarrow |z_2|^2 = 4 \Rightarrow z_2\overline{z_2} = 4 \Rightarrow \overline{z_2} = \frac{4}{z_2}$$

$$|z_3| = 3 \Rightarrow |z_3|^2 = 9 \Rightarrow z_3\overline{z_3} = 9 \Rightarrow \overline{z_3} = \frac{9}{z_3}$$

Show that: $|9z_1z_2+4z_1z_3+z_2z_3| = 6$

$$|z_1+z_2+z_3| = \left|\frac{1}{\overline{z_1}}+\frac{4}{\overline{z_2}}+\frac{9}{\overline{z_3}}\right|$$

$$1 = \left|\frac{\overline{z_2}\,\overline{z_3} + 4\overline{z_1}\,\overline{z_3} + 9\overline{z_1}\,\overline{z_2}}{\overline{z_1}\,\overline{z_2}\,\overline{z_3}}\right|$$

$$1 = \left|\frac{\overline{z_2z_3}+4\,\overline{z_1z_3}+9\,\overline{z_1z_2}}{\overline{z_1z_2z_3}}\right| \Rightarrow 1 = \frac{|\overline{z_2z_3}+4\,\overline{z_1z_3}+9\,\overline{z_1z_2}|}{|\overline{z_1z_2z_3}|}$$

$$|\overline{z_1z_2z_3}| = |\overline{z_2z_3}+4\,\overline{z_1z_3}+9\,\overline{z_1z_2}|$$

$|z| = |\overline{z}|$

$$|z_1z_2z_3| = |\overline{z_2z_3+4z_1z_3+9z_1z_2}|$$

$|z_1z_2| = |z_1||z_2|$

$$|z_1||z_2||z_3| = |z_2z_3|+|4z_1z_3|+|9z_1z_2|$$

$$1(2)(3) = |z_2z_3+4z_1z_3+9z_1z_2|$$

$$6 = |9z_1z_2+4z_1z_3+z_2z_3|$$

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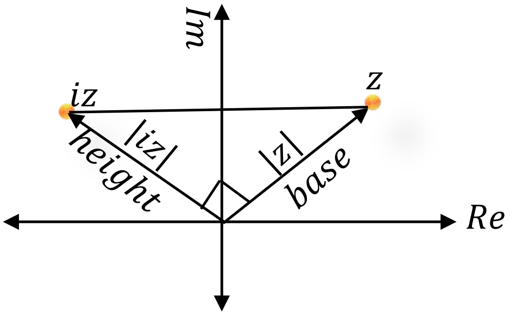
8. ***If the area of the triangle formed by the vertices z, iz and $z+iz$ is 50 Square units, find the value of $|z|$.***



$Area\ of\ a\ triangle = 50$

$\frac{1}{2} \times base \times height = 50$

$\frac{1}{2} \times |z| \times |iz| = 50$ $\boxed{|iz| = |z|}$

$\frac{1}{2} \times |z| \times |z| = 50 \implies \frac{1}{2} \times |z|^2 = 50$

$|z|^2 = 100 \implies |z| = \sqrt{100}$

$|z| = 10$

9. ***Show that the equation $z^3 + 2\bar{z} = 0$ has five solutions.***

$\boxed{|z^n| = |z|^n}$

$\boxed{|\bar{z}| = |z|}$

$z^3 + 2\bar{z} = 0$

$z^3 = -2\bar{z} \implies |z^3| = |-2\bar{z}|$

$|z|^3 = |-2||\bar{z}| \implies |z|^3 = 2|\bar{z}|$

$|z|^3 = 2|z|$

$|z|^3 - 2|z| = 0 \implies |z|(|z|^2 - 2) = 0$

$|z| = 0,\ |z|^2 - 2 = 0$

$z = 0,\ |z|^2 = 2 \implies z\bar{z} = 2 \implies \bar{z} = \frac{2}{z}$

$sub\ \bar{z} = \frac{2}{z}\ in\ z^3 + 2\bar{z} = 0$

$z^3 + 2\left(\frac{2}{z}\right) = 0 \implies z^3 + \frac{4}{z} = 0$

$z^3 = -\frac{4}{z} \implies z^4 = -4$

It has 4 non − zero solution and one zero solution, there are five solutions.

10. ***Find the square roots of (i) $4+3i$***

$$\sqrt{a+ib} = \pm \sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}$$

$Let\ z = 4 + 3i$

$|z| = |4+3i| = \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5$

Applying formula for square root.

$$\sqrt{4+3i} = \pm\left(\sqrt{\frac{5+4}{2}} + i\frac{b}{|b|}\sqrt{\frac{5-4}{2}}\right)$$

$\boxed{\frac{b}{|b|} = \frac{3}{|3|} = \frac{\cancel{3}}{\cancel{3}}\ 1}$

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$$= \pm\left(\sqrt{\frac{9}{2}} + (1)i\sqrt{\frac{1}{2}}\right) = \pm\left(\frac{3}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$(ii)\ -6 + 8i$

$$\sqrt{a+ib} = \pm\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}$$

$Let\ z = -6 + 8i$

$$|z| = |-6+8i| = \sqrt{(-6)^2 + (8)^2} = \sqrt{36+64} = \sqrt{100} = 10$$

Applying formula for square root.

$$\sqrt{-6+8i} = \pm\left(\sqrt{\frac{10+(-6)}{2}} + i\frac{b}{|b|}\sqrt{\frac{10-(-6)}{2}}\right)$$

$$\frac{b}{|b|} = \frac{8}{|8|} = \frac{8}{8} = 1$$

$$= \pm\left(\sqrt{\frac{10-6}{2}} + (1)i\sqrt{\frac{10+6}{2}}\right) = \pm\left(\sqrt{\frac{4}{2}} + i\sqrt{\frac{16}{2}}\right)$$

$$= \pm\left(\sqrt{2} + i\sqrt{8}\right)$$

$(iii)\ -5 - 12i$

$$\sqrt{a+ib} = \pm\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}$$

$Let\ z = -6 + 8i$

$$|z| = |-5-12i| = \sqrt{(-5)^2 + (-12)^2} = \sqrt{25+144} = \sqrt{169} = 13$$

Applying formula for square root.

$$\frac{b}{|b|} = \frac{-12}{|-12|} = -\frac{12}{12} = -1$$

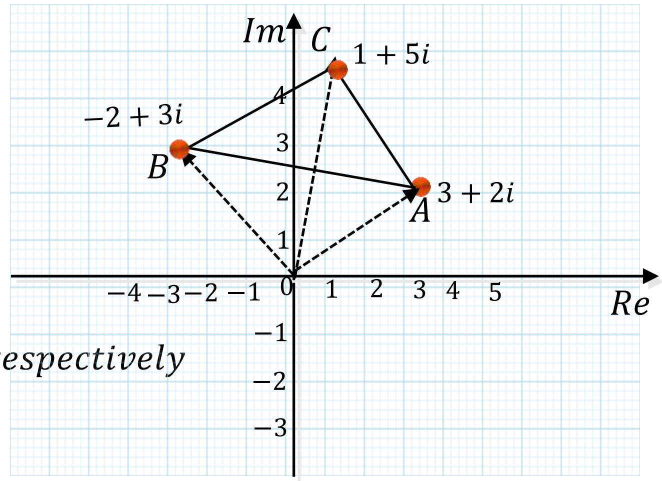
$$\sqrt{-5-12i} = \pm\left(\sqrt{\frac{13+(-5)}{2}} + \frac{b}{|b|}i\sqrt{\frac{13-(-5)}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{13-65}{2}} + (-1)i\sqrt{\frac{13+5}{2}}\right) = \pm\left(\sqrt{\frac{8}{2}} - i\sqrt{\frac{18}{2}}\right)$$

$$= \pm\left(\sqrt{4} - i\sqrt{9}\right) = \pm(2 - i3)$$

Exercise: 2.6

Example 2.18 Given the complex number $z = 3 + 2i$, represent the complex numbers z, iz and $z + iz$ and in one Argand diagram. Show that these complex numbers form the vertices of an isosceles right triangle.



Given : $z = 3 + 2i$

$$iz = i(3 + 2i) = 3i + 2i^2$$

$$= 3i + 2(-1)$$

$$iz = 3i - 2$$

$$z + iz = 3 + 2i + 3i - 2$$

$$z + iz = 1 + 5i$$

Let A, B and C be z, iz and z + iz respectively

$$AB = |B - A|$$

$$= |-2 + 3i - (3 + 2i)|$$

$$= |-2 + 3i - 3 - 2i| = |-5 + i|$$

$$= \sqrt{(-5)^2 + 1^2} = \sqrt{25 + 1} = \sqrt{26}$$

$$AB = \sqrt{26} \Longrightarrow AB^2 = 26$$

$$BC = |C - B|$$

$$= |1 + 5i - (-2 + 3i)| = |1 + 5i + 2 - 3i|$$

$$= |3 + 2i| = \sqrt{(3)^2 + 2^2} = \sqrt{9 + 4}$$

$$BC = \sqrt{13} \Longrightarrow BC^2 = 13$$

$$AC = |C - A|$$

$$= |1 + 5i - (3 + 2i)| = |1 + 5i - 3 - 2i|$$

$$= |-2 + 3i| = \sqrt{(-2)^2 + 3^2} = \sqrt{4 + 9}$$

$$AC = \sqrt{13} \Longrightarrow AC^2 = 13$$

$$AB^2 = BC^2 + AC^2$$

$BC = AC$, ΔABC *is an isosceles right triangle*

***Example 2.19**: show that $|3z - 5 + i| = 4$, represent a circle.*

$$|3z - 5 + i| = 4 \Longrightarrow 3\left|\frac{3z}{3} - \frac{5}{3} + \frac{i}{3}\right| = 4$$

$$3\left|z - \frac{5}{3} + \frac{i}{3}\right| = 4 \Longrightarrow \left|z - \frac{5}{3} + \frac{i}{3}\right| = \frac{4}{3} \Longrightarrow \left|z - \left(\frac{5}{3} - \frac{i}{3}\right)\right| = \frac{4}{3}$$

It is of the form $|z - z_0| = r$. *Hence it is a circle*

$$Centre = \left(\frac{5}{3}, \frac{1}{3}\right) \quad radius = \frac{4}{3}$$

Example 2.20: Show that $|z+2-i|<2$, represents interior points of a circle. Find its centre and radius.

Consider: $|z+2-i|=2$

$|z-(-2+i)|=2$

Centre $=-2+i=(-2,1)$

radius $=2$

$|z-(-2+i)|<2$, *represents the points of interior points of a cricle.*

Example 2.21: Oftain the cartesian form of the locus of z in each following (i) $|z|=|z-i|$ (ii) $|2z-3-i|=3$

(i) $|z|=|z-i|$

Let $z=x+iy$

$|x+iy|=|x+iy-i| \Rightarrow |x+iy|=|x+i(y-1)|$

$\sqrt{x^2+y^2}=\sqrt{x^2+(y-1)^2} \Rightarrow x^2+y^2=x^2+(y-1)^2$

Squaring on both sides.

$x^2+y^2=x^2+y^2-2y+1$

$x^2+y^2-x^2-y^2+2y-1=0$

$2y-1=0$

$\therefore$ *The locus is Straight line*

(ii) $|2z-3-i|=3$

Let $z=x+iy$

$|2(x+iy)-3-i|=3$

$|2x+2iy-3-i|=3 \Rightarrow |2x-3+2iy-i|=3$

$|2x-3+i(2y-1)|=3$

$\sqrt{(2x-3)^2+(2y-1)^2}=3 \Rightarrow (2x-3)^2+(2y-1)^2=9$

Squaring on both sides.

$(2x)^2-2(2x)(3)+(3)^2+(2y)^2-2(2y)(1)+(1)^2=9$

$4x^2-12x+9+4y^2-4y+1$

$4x^2+4y^2-12x-4y+10=9$

$4x^2+4y^2-12x-4y+10-9=0$

$4x^2+4y^2-12x-4y+1=0$ $\therefore$ *The locus is a cricle.*

1. If $z=x+iy$ is a complex number such that $\left|\frac{z-4i}{z+4i}\right|=1$. Show that the locus of z is real axis.

$z=x+iy$

$\left|\frac{z-4i}{z+4i}\right|=1 \Rightarrow \left|\frac{x+iy-4i}{x+iy+4i}\right|=1$

$$\left|\frac{x+i(y-4)}{x+i(y+4)}\right|=1 \Rightarrow \frac{|x+i(y-4)|}{|x-i(y+4)|}=1$$

$$\frac{\sqrt{x^2+(y-4)^2}}{\sqrt{x^2+(y+4)^2}}=1 \Rightarrow \sqrt{x^2+(y-4)^2}=\sqrt{x^2+(y+4)^2}$$

Squaring on both sides.

$$x^2+(y-4)^2=x^2+(y+4)^2$$

$$x^2+y^2-2(y)(4)+4^2=x^2+y^2-2(y)(4)+4^2$$

$$x^2+y^2-8y+16=x^2+y^2-8y+16$$

$$x^2+y^2-8y+16-x^2-y^2-8y-16=0$$

$$-8y-8y=0 \Rightarrow -16y=0$$

$$\boxed{y=0}$$

$\therefore$ *The locus of Z is real axis or x − axis.*

2. If $z=x+iy$ is a complex number such that $Im\left(\frac{2z+1}{iz+1}\right)=0$. Show that the locus of z is $2x^2+2y^2+x-2=0$.

$$Given: Im\left(\frac{2z+i}{iz+1}\right)=0$$

$$Take: \frac{2z+i}{iz+1}=\frac{2(x+iy)+1}{i(x+iy)+1}=\frac{2x+i2y+1}{ix+\underset{(-1)}{i^2}y+1}=\frac{2x+1+i2y}{-y+1+ix}$$

$$=\frac{(2x+1)+i2y}{(1-y)+ix}\times\frac{(1-y)-ix}{(1-y)-ix}$$

$$=\frac{(2x+1)(1-y)-ix(2x+1)+i2y(1-y)-\overset{(-1)}{i^2}2xy}{(1-y)^2+x^2}$$

$$=\frac{(2x+1)(1-y)-ix(2x+1)+i2y(1-y)+2xy}{(1-y)^2+x^2}$$

$$=\frac{(2x+1)(1-y)+2xy+i[2y(1-y)-x(2x+1)]}{(1-y)^2+x^2}$$

$$=\frac{(2x+1)(1-y)+2xy}{(1-y)^2+x^2}+\frac{i[2y(1-y)-x(2x+1)]}{(1-y)^2+x^2}$$

$$Im\left(\frac{2z+1}{iz+1}\right)=0$$

$$\frac{2y(1-y)-x(2x+1)}{(1-y)^2+x^2}=0 \Rightarrow 2y(1-y)-x(2x+1)=0$$

$$2y-2y^2-2x^2-x=0 \Rightarrow -2x^2-2y^2-x+2y=0$$

$$2x^2+2y^2+x-2y=0$$

3. Obtain the cartesian form of the locus of $z = x + iy$ in each of the following cases.

(i) $[Re(iz)]^2 = 3$

$iz = i(x+iy) = ix + i^2y$

$= ix - y$

$iz = -y + ix$

$Re(iz) = -y$

$[Re(iz)]^2 = 3 \Rightarrow [-y]^2 = 3$

$y^2 = 3$

(ii) $Im[(1-i)z+1] = 0$

Let $z = x + iy$

$(1-i)z + 1 = (1-i)(x+iy) + 1$

$= x + iy - ix - i^2y + 1 = x + iy - ix + y + 1$

$= x + y + i(y-x) + 1$

$(1-i)z + 1 = x + y + 1 + i(y-x)$

$Im[(1-i)z+1] = 0$

$y - x = 0 \Rightarrow -x + y = 0 \Rightarrow x - y = 0$

(iii) $|z+i| = |z-1|$

Let $z = x + iy$

$|x + iy + i| = |x + iy - 1|$

$|x + i(y+1)| = |(x-1) + iy|$

$\sqrt{x^2 + (y+1)^2} = \sqrt{(x-1)^2 + y^2}$

Squaring on both sides.

$x^2 + (y+1)^2 = (x-1)^2 + y^2$

$x^2 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2$

$x^2 + y^2 + 2y + 1 - x^2 + 2x - 1 - y^2 = 0$

$2y + 2x = 0$

$\div$ by 2

$x + y = 0$

(iv) $\bar{z} = z^{-1}$

$\bar{z} = \frac{1}{z} \Rightarrow z\bar{z} = 1$

$|z|^2 = 1$

$|x+iy|^2 = 1 \Rightarrow \left(\sqrt{x^2+y^2}\right)^2 = 1$

$x^2 + y^2 = 1$

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4. *Show that the following equations represent a circle, find its centre and radius.*

$(\boldsymbol{i})|\boldsymbol{z-2-i}|=\mathbf{3}$

$|z-(2+i)|=3$

it is of the form $|z-z_0|=r$. *Hence it is a circle*

$Centre=(2+i)=(2,1)$

$radius=3$

$(ii)|2z+2-4i|=2$

$|2z+2-4i|=2 \implies 2|z+1-2i|=2$

$|z+1-2i|=\frac{2}{2} \implies |z+1-2i|=1$

$|z-(-1+2i)|=1$

it is of the form $|z-z_0|=r$. *Hence it is a circle*

$Centre=(-1+2i)=(-1,2)$

$radius=1$

$(\boldsymbol{iii})|\boldsymbol{3z-6+12i}|=\mathbf{8}$

$|3z-6+12i|=8 \implies 3|z-2+4i|=8$

$|z-2+4i|=\frac{8}{3} \implies |z-(2-4i)|=\frac{8}{3}$

it is of the form $|z-z_0|=r$. *Hence it is a circle*

$Centre=2-4i=(2,-4)$

$radius=\frac{8}{3}$

5. *Obtain the cartesian equation for the locus of* $\boldsymbol{z=x+iy}$ ***in each of the following cases*:**

$(\boldsymbol{i})|\boldsymbol{z-4}|=\mathbf{16}$

Let $z=x+iy$

$|x+iy-4|=16 \implies |x-4+iy|=16$

$\sqrt{(x-4)^2+y^2}=16 \implies (x-4)^2+y^2=(16)^2$

Squaring on both sides.

$x^2-2(x)(4)+4^2+y^2=256 \implies x^2-8x+16+y^2=256$

$x^2-8x+y^2+16=256 \implies x^2+y^2-8x+16-256=0$

$x^2+y^2-8x-240=0$ *Which is a cricle.*

$(\boldsymbol{ii})|\boldsymbol{z-4}|^2-|\boldsymbol{z-1}|^2=\mathbf{16}$

Let $z=x+iy$

$|x + iy - 4|^2 - |x + iy - 1|^2 = 16$

$|(x-4) + iy|^2 - |(x-1) + iy|^2 = 16$

$\left(\sqrt{(x-4)^2 + y^2}\right)^2 - \left(\sqrt{(x-1)^2 + y^2}\right)^2 = 16$

$(x-4)^2 + y^2 - [(x-1)^2 + y^2] = 16$

$x^2 - 8x + 16 + y^2 - [x^2 - 2x + 1 + y^2] = 16$

$x^2 - 8x + 16 + y^2 - x^2 + 2x - 1 - y^2 = 16$

$-6x + 15 = 16 \Longrightarrow -6x + 15 - 16 = 0$

$-6x - 1 = 0 \Longrightarrow 6x + 1 = 0$ $\therefore$ *The locus is Straight line*

THEORY OF EQUATION − EXERCISE 3.1

Example 3.1 If α and β are the roots of $17x^2+43x-73=0$, then form a equation whose roots are α + 2 and β + 2

If α and β are the roots of $17x^2+43x-73=0$

Let $a=17, b=43, c=-73$

Sum of the roots : $\alpha+\beta=-\frac{b}{a}=-\frac{43}{17}$

$$\boxed{\alpha+\beta=-\frac{43}{17}}$$

product of the roots : $\alpha\beta=\frac{c}{a} \Rightarrow \boxed{\alpha\beta=-\frac{73}{17}}$

To form a Quadratic equation whose roots are $\alpha+2$ *and* $\beta+2$

Sum of the roots $=\alpha+2+\beta+2$

$$=\alpha+\beta+4$$

$$=-\frac{43}{17}+4=\frac{-43+68}{17}=\frac{25}{17}$$

$$\boxed{\text{Sum of the roots }=\frac{25}{17}}$$

product of the roots $=(\alpha+2)(\beta+2)$

$$=\alpha\beta+2\alpha+2\beta+4=\alpha\beta+2(\alpha+\beta)+4$$

$$=-\frac{73}{17}+2\left(-\frac{43}{17}\right)+4=-\frac{73}{17}-\frac{86}{17}+4$$

$$=\frac{-73-86}{17}+4=\frac{-159}{17}+4=\frac{-159+68}{17}$$

$$\boxed{\text{product of the roots }=-\frac{91}{17}}$$

Quadratic equation : $x^2-(\text{sum of the roots})x+\text{product of the roots}=0$

$$x^2-\frac{25}{17}x-\frac{91}{17}=0$$

multiply by 17 on both sides

$$17x^2-25x-91=0$$

2. If α and β are the roots of $2x^2-3x-5=0$, form a equation whose roots are α^2 and β^2.

If α and β are the roots of $2x^2-3x-5=0$

$a=2, b=-3, c=-5$

Sum of the roots : $\alpha+\beta=-\frac{b}{a} \Longrightarrow \alpha+\beta=\frac{3}{2}$

product of the roots : $\alpha\beta=\frac{c}{a} \Longrightarrow \alpha\beta=-\frac{5}{2}$

To form a Quadratic equation whose roots are α^2 *and* β^2

Sum of the roots $=\alpha^2+\beta^2$

$$=(\alpha+\beta)^2-2\alpha\beta$$

$$=\left(\frac{3}{2}\right)^2-2\left(-\frac{5}{2}\right)=\frac{9}{4}+5$$

$$=\frac{9+20}{4}=\frac{29}{4}$$

$(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta$
$(\alpha+\beta)^2-2\alpha\beta=\alpha^2+\beta^2$

$\therefore$ Sum of the roots : $=\frac{29}{4}$

product of the roots $=\alpha^2\beta^2$

$$=(\alpha\beta)^2=\left(\frac{-5}{2}\right)^2=\frac{25}{4}$$

$\therefore$ product of the roots $=\frac{25}{4}$

Quadratic equation: $x^2-(sum\ of\ the\ roots)x+product\ of\ the\ roots=0$

$$x^2-\frac{29}{4}x+\frac{25}{4}=0$$

multiply by 4 on both sides

$$4x^2-4\times\frac{29}{4}x+4\times\frac{25}{4}=0$$

$$\boxed{4x^2-29x+25=0}$$

Example 3.4*: *Find the sum of the squares of the roots of $\boldsymbol{ax^4+bx^3+cx^2+dx+e=0}$

Let α,β,γ *and* δ *be the roots of* $ax^4+bx^3+cx^2+dx+e=0$

$$\Sigma_1=\alpha+\beta+\gamma+\delta=-\frac{b}{a}$$

$$\Sigma_2=\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}$$

$$\Sigma_3=\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}$$

$$\Sigma_4=\alpha\beta\gamma\delta=\frac{e}{a}$$

To find $\alpha^2+\beta^2+\gamma^2+\delta^2$

$$\alpha^2+\beta^2+\gamma^2+\delta^2 \equiv (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$$

$$=\left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2-2ac}{a^2}$$

$$(a+b+c+d)^2 \equiv a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$$

$$\equiv a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)$$

$$(a+b+c+d)^2 - 2(ab+ac+ad+bc+bd+cd) = a^2+b^2+c^2+d^2$$

Example 3.5*: *Find the condition that the roots of $x^3+ax^2+bx+c=0$ are in the ratio $p:q:r$

$$x^3+ax^2+bx+c=0$$

$$a=1, b=a, c=b, d=c$$

Given roots are in the ratio $p:q:r$

. Assume the roots as $p\lambda, q\lambda$ and $r\lambda$

$$\Sigma_1 = p\lambda+q\lambda+r\lambda = -\frac{b}{a} = -\frac{a}{1} = -a$$

$$p\lambda+q\lambda+r\lambda = -a$$

$$\lambda(p+q+r) = -a \implies \lambda = \frac{-a}{p+q+r}$$

$$\Sigma_3 = (p\lambda)(q\lambda)(r\lambda) = -\frac{d}{a} = -\frac{c}{1} = -c$$

$$(p\lambda)(q\lambda)(r\lambda) = -c$$

$$\lambda^3 pqr = -c \implies \lambda^3 = \frac{-c}{pqr}$$

$$sub\ \lambda = \frac{-a}{p+q+r}$$

$$\left(\frac{-a}{p+q+r}\right)^3 = \frac{-c}{pqr} \implies \frac{-a^3}{(p+q+r)^3} = \frac{-c}{pqr}$$

$$a^3pqr = c(p+q+r)^3$$

Example 3.6*: *Form the equation whose roots are the squares of the roots of the cubic equation $x^3+ax^2+bx+c=0$

Let α,β,γ be the roots of $x^3+ax^2+bx+c=0$

$$a=1, b=a, c=b, d=c$$

$$\Sigma_1 = \alpha+\beta+\gamma = -\frac{b}{a} = \frac{-a}{1} = -a \implies \alpha+\beta+\gamma = -a$$

$$\Sigma_2 = \alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a} = \frac{b}{1} = b \implies \boxed{\alpha\beta+\beta\gamma+\gamma\alpha = b}$$

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$$\Sigma_3 = \alpha\beta\gamma = -\frac{d}{a} = -c \Rightarrow \boxed{\alpha\beta\gamma = -c}$$

To form the equation whose roots are $\alpha^2, \beta^2, \gamma^2$

$$\Sigma_1 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$
$$(a + b + c)^2 - 2(ab + bc + ca) = a^2 + b^2 + c^2$$

$$\Sigma_1 = (-a)^2 - 2b \Rightarrow \Sigma_1 = a^2 - 2b$$

$$\Sigma_2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2$$

$$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2[(\alpha\beta)(\beta\gamma) + (\beta\gamma)(\gamma\alpha) + (\gamma\alpha)(\alpha\beta)]$$

$$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma[\beta + \gamma + \alpha]$$

$$= b^2 - 2(-c)(-a) = b^2 - 2ca$$

$$\Sigma_2 = b^2 - 2ca$$

$$\Sigma_3 = \alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = (-c)^2 = c^2$$

$$\Sigma_3 = c^2$$

Hence the required equation : $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$$x^3 - (a^2 - 2b)x^2 + (b^2 - 2ca)x - c^2 = 0$$

Example 3.7 : If p is real, discuss the nature of the roots of the equation $4x^2 + 4px + p + 2 = 0$ in terms of p

$$4x^2 + 4px + p + 2 = 0$$

$$a = 4, \quad b = 4p, \quad c = p + 2$$

Disciminant : $\Delta = b^2 - 4ac$

$$\Delta = (4p)^2 - 4(4)(p + 2)$$
$$= 16p^2 - 16(p + 2)$$
$$= 16p^2 - 16p - 32 = 16(p^2 - p - 2)$$
$$= 16(p + 1)(p - 2)$$

$\Delta < 0$ *if* $-1 < p < 2$ (*Imaginary roots*)

$\Delta = 0$ *if* $p = -1$ *or* $p = 2$ (*equal roots*)

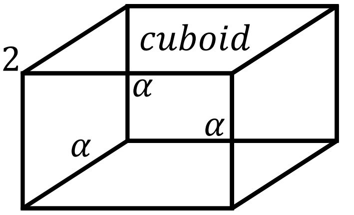
$\Delta > 0$ *if* $-\infty < p < -1$ *or* $2 < p < \infty$ (*distinct real roots*)

EXERCISE 3.1

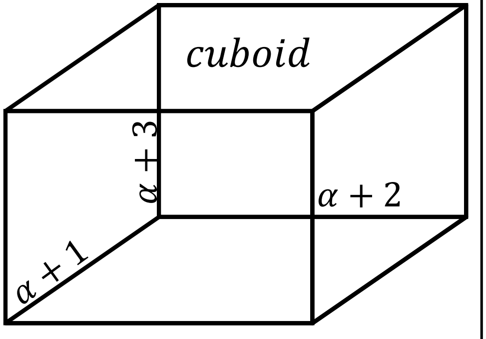
1. *If the sides of a cubic box are increased by 1, 2, 3 units respectively to form a cuboid, then the volume is increased by 52 cubic units. Find the volume of the cuboid.*

let α *be the side of a cube then its volume is* α^3

If the sides of cube is increased by 1, 2, 3 *units it becomes cuboid then volume of cuboid is* $\alpha^3 + 52$



Sides of a cuboid are $(\alpha+1), (\alpha+2), (\alpha+3)$



$$\text{volume of cuboid} = \alpha^3 + 52$$

$$(\alpha+1)(\alpha+2)(\alpha+3) = \alpha^3 + 52$$

$$(\alpha^2 + 2\alpha + \alpha + 2)(\alpha+3) = \alpha^3 + 52$$

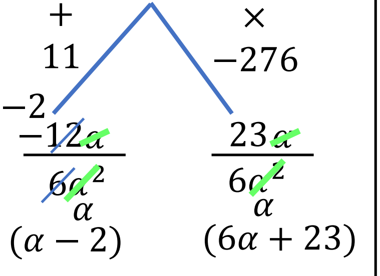
$$(\alpha^2 + 3\alpha + 2)(\alpha+3) = \alpha^3 + 52$$

$$\alpha^3 + 3\alpha^2 + 2\alpha + 3\alpha^2 + 9\alpha + 6 = \alpha^3 + 52$$

$$\cancel{\alpha^3} + 6\alpha^2 + 11\alpha + 6 = \cancel{\alpha^3} + 52$$

$$6\alpha^2 + 11\alpha + 6 - 52 = 0$$

$$6\alpha^2 + 11\alpha - 46 = 0$$



$$(\alpha - 2)(6\alpha + 23) = 0$$

$$\alpha - 2 = 0, \quad 6\alpha + 23 = 0$$

$$\alpha = 2, \quad 6\alpha = -23$$

$$\boxed{\alpha = -\frac{23}{6}}$$

Volume of cube $= \alpha^3 = 2^3$

Volume of cube $= 8$ *cubic units*

Volume of cuboid $= \alpha^3 + 52$

Volume of cuboid $= 8 + 52 \Longrightarrow$ *Volume of cuboid* $= 60$

2. construct a cubic equation with roots (*i*) 1, 2 *and* 3

Let $\alpha = 1, \beta = 2, \gamma = 3$

$$\Sigma_1 = \alpha + \beta + \gamma = 1 + 2 + 3$$

$$\Sigma_1 = 6$$

$$\Sigma_2 = \alpha\beta + \beta\gamma + \gamma\alpha$$

$$= (1)(2) + 2(3) + (3)(1) = 2 + 6 + 3$$

$$\Sigma_2 = 11$$

$$\Sigma_3 = \alpha\beta\gamma = (1)(2)(3)$$

$$\Sigma_3 = 6$$

Required equation : $x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0$

or

$$x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$$

$$x^3 - 6x^2 + 11x - 6 = 0$$

(ii) 1, 1 *and* -2

Let $\alpha = 1, \beta = 1, \gamma = -2$

$\Sigma_1 = \alpha + \beta + \gamma = 1 + 1 - 2$

$\Sigma_1 = 0$

$\Sigma_2 = \alpha\beta + \beta\gamma + \gamma\alpha = (1)(1) + 1(-2) + (-2)(1)$

$= 1 - 2 - 2 = 1 - 4$

$\Sigma_2 = -3$

$\Sigma_3 = \alpha\beta\gamma = (1)(1)(-2)$

$\Sigma_3 = -2$

Required equation : $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$x^3 - 0x^2 - 3x + 2 = 0$

$x^3 - 3x + 2 = 0$

(iii) 2, −2 and 4

Let $\alpha = 2, \beta = -2, \gamma = 4$

$\Sigma_1 = \alpha + \beta + \gamma = 2 - 2 + 4$

$\Sigma_1 = 4$

$\Sigma_2 = \alpha\beta + \beta\gamma + \gamma\alpha = (2)(-2) + (-2)(4) + 4(2)$

$= -4 - 8 + 8$

$\Sigma_2 = -4$

$\Sigma_3 = \alpha\beta\gamma = 2(-2)(4)$

$\Sigma_3 = -16$

Required equation : $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$x^3 - 4x^2 - 4x + 16 = 0$

3. *If* α, β, γ *are the roots of the cubic equation* $x^3 + 2x^2 + 3x + 4 = 0$, *form a cubic equation whose roots are*

(i) $2\alpha, 2\beta, 2\gamma$ **(ii)** $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ **(iii)** $-\alpha, -\beta, -\gamma$

Let α, β, γ *are the roots of* $x^3 + 2x^2 + 3x + 4 = 0$

$a = 1, b = a, c = b, d = c$

$\Sigma_1 = \alpha + \beta + \gamma = -\frac{b}{a} = \frac{-2}{1} \implies \alpha + \beta + \gamma = -2$

$\Sigma_2 = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 3 \implies \alpha\beta + \beta\gamma + \gamma\alpha = 3$

$\Sigma_3 = \alpha\beta\gamma = -\frac{d}{a} = -4 \implies \alpha\beta\gamma = -4$

(i) $2\alpha, 2\beta, 2\gamma$

$\Sigma_1 = 2\alpha + 2\beta + 2\gamma = 2(\alpha + \beta + \gamma) = 2(-2)$

$\boxed{\Sigma_1 = -4}$

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$\Sigma_2 = (2\alpha)(2\beta) + (2\beta)(2\gamma) + (2\gamma)(2\alpha) = 4\alpha\beta + 4\beta\gamma + 4\gamma\alpha$

$= 4(\alpha\beta + \beta\gamma + \gamma\alpha) = 4(3)$

$\boxed{\Sigma_2 = 12}$

$\Sigma_3 = (2\alpha)(2\beta)(2\gamma) = 8(\alpha\beta\gamma) = 8(-4)$

$\boxed{\Sigma_3 = -32}$

Cubic equation : $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$x^3 - (-4)x^2 + 12x + 32 = 0$

$x^3 + 4x^2 + 12x + 32 = 0$

(ii) $\boldsymbol{\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}}$

$$\Sigma_1 = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{3}{-4}$$

$\boxed{\Sigma_1 = -\frac{3}{4}}$

$$\Sigma_2 = \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) + \left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right) + \left(\frac{1}{\gamma}\right)\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} = \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma}$$

$$= \frac{\alpha + \beta + \gamma}{\alpha\beta\gamma} = \frac{-2}{-4}$$

$\boxed{\Sigma_2 = \frac{1}{2}}$

$$\Sigma_3 = \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right) = \frac{1}{\alpha\beta\gamma} = \frac{1}{-4}$$

$$\Sigma_3 = -\frac{1}{4}$$

Cubic equation : $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$$x^3 + \frac{3}{4}x^2 + \frac{1}{2}x + \frac{1}{4} = 0$$

multiplying both side by 4

$$4x^3 + 3x^2 + 2x + 1 = 0$$

(iii) $-\alpha, -\beta, -\gamma$

$\Sigma_1 = -\alpha - \beta - \gamma = -(\alpha + \beta + \gamma) = -(-2)$

$\Sigma_1 = 2$

$\Sigma_2 = (-\alpha)(-\beta) + (-\beta)(-\gamma) + (-\gamma)(-\alpha)$

$= \alpha\beta + \beta\gamma + \gamma\alpha$

$\Sigma_2 = 3$

$\Sigma_3 = -\alpha \times -\beta \times -\gamma = -\alpha\beta\gamma = -(-4)$

$\Sigma_3 = 4$

$Cubic\ equation : x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$x^3 - 2x^2 + 3x - 4 = 0$

4. *Solve the equation* $3x^3 - 16x^2 + 23x - 6 = 0$ *if the product of two roots is* 1

$Let\ \alpha, \beta, \gamma\ are\ the\ roots\ of\ 3x^3 - 16x^2 + 23x - 6 = 0$

$a = 3, b = -16, c = 23, d = -6$

$Given\ product\ of\ two\ roots = 1$

$\alpha\beta = 1$

$\Sigma_1\alpha = \alpha + \beta + \gamma = -\frac{b}{a} = -\left(\frac{-16}{3}\right) = \frac{16}{3} \Rightarrow \alpha + \beta + \gamma = \frac{16}{3} \ldots (1)$

$\Sigma_2\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{23}{3} \Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = \frac{23}{3} \ldots (2)$

$\Sigma_3\alpha\beta\gamma = \alpha\beta\gamma = -\frac{d}{a} = \frac{6}{3} = 2 \Rightarrow \alpha\beta\gamma = 2 \ldots (3)$

$sub\ \alpha\beta = 1\ in\ (3)$

$1 \times \gamma = 2 \Rightarrow \gamma = 2$

$sub\ \gamma = 2\ in\ (1)\ \alpha + \beta + \gamma = \frac{16}{3}$

$\alpha + \beta + 2 = \frac{16}{3} \Rightarrow \alpha + \beta = \frac{16}{3} - 2$

$\alpha + \beta = \frac{16 - 6}{3} \Rightarrow \alpha + \beta = \frac{10}{3}$

$\beta = \frac{10}{3} - \alpha\ in\ \alpha\beta = 1$

$\alpha\left(\frac{10}{3} - \alpha\right) = 1 \Rightarrow \alpha\left(\frac{10 - 3\alpha}{3}\right) = 1 \Rightarrow \alpha(10 - 3\alpha) = 3$

$10\alpha - 3\alpha^2 = 3 \Rightarrow 10\alpha - 3\alpha^2 - 3 = 0 \Rightarrow -3\alpha^2 + 10\alpha - 3 = 0$

$3\alpha^2 - 10\alpha + 3 = 0 \implies (3\alpha - 1)(\alpha - 3) = 0$

$+ \quad \times$

$-10 \quad 9$

-3

$\frac{-1\alpha}{3\alpha^2} \quad \frac{-9\alpha}{3\alpha^2}$

$\alpha \quad \alpha$

$(3\alpha - 1) \quad (\alpha - 3)$

$$3\alpha - 1 = 0\,, \quad \alpha - 3 = 0$$

$$3\alpha = 1\,, \quad \alpha = 3$$

$$\alpha = \frac{1}{3}$$

$sub\ \alpha = 3\ in\ \alpha\beta = 1$

$$3\beta = 1 \implies \beta = \frac{1}{3}$$

$$\alpha = 3, \beta = \frac{1}{3}, \gamma = 2$$

5. *Find the sum of squares of roots of the equation* $2x^4 - 8x^3 + 6x^2 - 3 = 0$

Let $\alpha, \beta, \gamma, \delta$ *be the roots of the equation* $2x^4 - 8x^3 + 6x^2 - 3 = 0$

$$a = 2, b = -8, c = 6, d = 0, e = -3$$

$$\Sigma_1\alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a} = -\left(\frac{-8}{2}\right)$$

$$\alpha + \beta + \gamma + \delta = 4$$

$$\Sigma_2\alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a} = \frac{6}{2}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 3$$

$$\boxed{(a+b+c+d)^2 \equiv a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd)}$$

$$\boxed{(a+b+c+d)^2 - 2(ab + ac + ad + bc + bd + cd) \equiv a^2 + b^2 + c^2 + d^2}$$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$$= 4^2 - 2(3) = 16 - 6 = 10$$

$$\boxed{\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 10}$$

6. *Solve the equation* $x^3 - 9x^2 + 14x + 24 = 0$ *if it is given that two of it roots are in the ratio* 3 : 2

Let α, β, γ *are the roots of the equation* $x^3 - 9x^2 + 14x + 24 = 0$

$$a = 1, b = -9, c = 14, d = 24$$

$$\alpha : \beta = 3 : 2 \implies \alpha = 3k, \ \beta = 2k$$

$$\Sigma_1\alpha = \alpha + \beta + \gamma = -\frac{b}{a} = -\left(\frac{-9}{1}\right) = 9$$

$$\alpha + \beta + \gamma = 9 \implies 3k + 2k + \gamma = 9$$

$$5k + \gamma = 9 \implies \gamma = 9 - 5k$$

$$\Sigma_2 = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 14$$

$$\boxed{\alpha\beta + \beta\gamma + \gamma\alpha = 14}$$

$$(3k)(2k) + (2k)(9 - 5k) + (9 - 5k)3k = 14$$

$$6k^2 + 18k - 10k^2 + 27k - 15k^2 = 14$$

$$-19k^2 + 45k - 14 = 0$$

$$19k^2 - 45k + 14 = 0$$

$+$, $\times$: -45 , 266 ; -2

$\frac{-38k}{19k^2} = \frac{-2}{k} \Rightarrow (k-2)$, $\frac{-7k}{19k^2} = \frac{-7}{19k} \Rightarrow (19k-7)$

$$(k - 2)(19k - 7) = 0$$

$$k - 2 = 0,\ 19k - 7 = 0$$

$$k = 2\ , \qquad 19k = 7 \Rightarrow k = \frac{7}{19}$$

$$\alpha = 3k,\ \beta = 2k,\ \gamma = 9 - 5k$$

$$when\ k = 2 \Rightarrow \alpha = 3(2),\ \beta = 2(2),\ \gamma = 9 - 5(2)$$

$$\alpha = 6\ ,\ \beta = 4\ ,\ \gamma = -1$$

$$when\ k = \frac{7}{19} \Rightarrow \alpha = 3\left(\frac{7}{19}\right),\ \beta = 2\left(\frac{7}{19}\right),\ \gamma = 9 - 5\left(\frac{7}{19}\right)$$

$$\alpha = \frac{21}{19},\ \beta = \frac{14}{19},\ \gamma = 9 - \frac{35}{19} \Rightarrow \gamma = \frac{171 - 35}{19} = \frac{136}{19}$$

$$\alpha = \frac{21}{19},\ \beta = \frac{14}{19},\ \gamma = \frac{136}{19}$$

7. *If* $\boldsymbol{\alpha, \beta, \gamma}$ *are the roots of the polynomial equations* $\boldsymbol{ax^3 + bx^2 + cx + d = 0}$ *value of* $\boldsymbol{\sum \frac{\alpha}{\beta\gamma}}$ *in terms of the coefficients*

Let α, β, γ *are the roots of the equation* $ax^3 + bx^2 + cx + d = 0$

$$\Sigma_1\alpha = \alpha + \beta + \gamma = \frac{-b}{a} \Rightarrow \alpha + \beta + \gamma = -\frac{b}{a}$$

$$\Sigma_2\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\Sigma_3 = \alpha\beta\gamma = \frac{-d}{a} \Rightarrow \alpha\beta\gamma = -\frac{d}{a}$$

$$To\ find \sum \frac{\alpha}{\beta\gamma} = \frac{\alpha}{\beta\gamma} + \frac{\beta}{\alpha\gamma} + \frac{\gamma}{\alpha\beta} = \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha\beta\gamma}$$

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

squaring on both sides

$$(\alpha + \beta + \gamma)^2 = \left(\frac{-b}{a}\right)^2 \Rightarrow \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) = \frac{b^2}{a^2}$$

$$where\ \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

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$$\alpha^2+\beta^2+\gamma^2+2\left(\frac{c}{a}\right)=\frac{b^2}{a^2} \Longrightarrow \alpha^2+\beta^2+\gamma^2+\frac{2c}{a}=\frac{b^2}{a^2}$$

$$\alpha^2+\beta^2+\gamma^2=\frac{b^2}{a^2}-\frac{2c}{a} \Longrightarrow \alpha^2+\beta^2+\gamma^2=\frac{b^2-2ac}{a^2}$$

$$\sum\frac{\alpha}{\beta\gamma}=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}=\frac{\frac{b^2-2ac}{a^2}}{\frac{-d}{a}}=\frac{b^2-2ac}{a^2}\times-\frac{a}{d}$$

$$\sum\frac{\alpha}{\beta\gamma}=\frac{2ac-b^2}{ad}$$

8. If α,β,γ and δ are the roots of the polynomial equation $2x^4+5x^3-7x^2+8=0$ find a quadratic equation with integer coefficient whose roots are $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma\delta$

$\alpha,\beta,\gamma,\delta$ be the roots of the equation $2x^4+5x^3-7x^2+0x+8=0$

$$a=2, b=5, c=-7, d=0, e=8$$

$$\Sigma_1=\alpha+\beta+\gamma+\delta=-\frac{b}{a}=-\frac{5}{2} \Longrightarrow \alpha+\beta+\gamma+\delta=-\frac{5}{2}$$

$$\Sigma_4=\alpha\beta\gamma\delta=-\frac{e}{a}=-\frac{8}{2}=-4 \Longrightarrow \alpha\beta\gamma\delta=-4$$

To form a quadratic equation whose roots are $-\frac{5}{2},-4$

$$\text{Sum of the roots}=-\frac{5}{2}-4=\frac{-5-8}{2}=-\frac{13}{2}$$

$$\text{product of the roots}=-\frac{5}{2}\times-4=10$$

$$x^2-(\text{sum of the roots})x+\text{product of the roots}=0$$

$$x^2-\left(-\frac{13}{2}\right)x+10=0 \Longrightarrow x^2+\frac{13}{2}x+10=0$$

$$\times 2$$

$$2x^2+13x+20=0$$

9. If p and q are the roots of the equation $lx^2+nx+n=0$ show that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0$

p and q are the roots of the equation $lx^2+nx+n=0$

$$\text{sum of the roots}: \ p+q=-\frac{n}{l} \ \dots(1)$$

$$\text{product of the roots}: \ pq=\frac{n}{l} \ \dots(2)$$

$$L.H.S = \sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} \;\; = \frac{\sqrt{p}}{\sqrt{q}} + \frac{\sqrt{q}}{\sqrt{p}} + \frac{\sqrt{n}}{\sqrt{l}}$$

$$= \frac{\sqrt{p}\sqrt{p}\sqrt{l} \; + \sqrt{q}\sqrt{q}\sqrt{l} \; + \sqrt{n}\sqrt{p}\sqrt{q}}{\sqrt{p} \times \sqrt{q} \times \sqrt{l}}$$

$$= \frac{p\sqrt{l} + q\sqrt{l} + \sqrt{pqn}}{\sqrt{pql}} \;\; = \frac{(p+q)\sqrt{l} + \sqrt{pqn}}{\sqrt{pql}}$$

$$p + q = -\frac{n}{l} \text{ and } pq = \frac{n}{l}$$

$$= \frac{\frac{-n}{l} \times \sqrt{l} + \sqrt{\frac{n}{l} \times n}}{\sqrt{\frac{n}{l} \times l}} \;\; = \frac{\frac{-n}{\sqrt{l} \times \sqrt{l}} \times \sqrt{l} + \sqrt{\frac{n^2}{l}}}{\sqrt{n}} \;\; = \frac{\frac{-n}{\sqrt{l}} + \frac{n}{\sqrt{l}}}{\sqrt{n}} \;\; = 0$$

10. *If the equation $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root. Show that it must be equal to $\frac{pq' - p'q}{q - q'}$ or $\frac{q - q'}{p' - p}$*

Let α be a common roots of $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$

$$\alpha^2 + p\alpha + q = 0 \dots (1) \text{ and } \alpha^2 + p'\alpha + q' = 0 \dots (2)$$

solve (1) and (2)

$$\alpha^2 + p\alpha + q = 0$$
$$(-) \quad (-) \quad (-)$$
$$\alpha^2 + p'\alpha + q' = 0$$

$$p\alpha - p'\alpha + q - q' = 0$$

$$\alpha(p - p') = q' - q$$

$$\alpha = \frac{q' - q}{p - p'} \implies \alpha = \frac{-(q - q')}{-(p' - p)} \implies \alpha = \frac{q - q'}{p' - p}$$

11. *Formalate into a mathematical problem to find a number such that when its cube root is added to it, the result is 6 .*

Let x be a number

Cube root of a number is added to a number = 6

$$\sqrt[3]{x} + x = 6$$

$$x^{1/3} + x = 6 \implies x^{1/3} = 6 - x$$

$$x = (6 - x)^3$$

$$x = 6^3 - 3(6)^2(x) + 3(6)x^2 - x^3$$

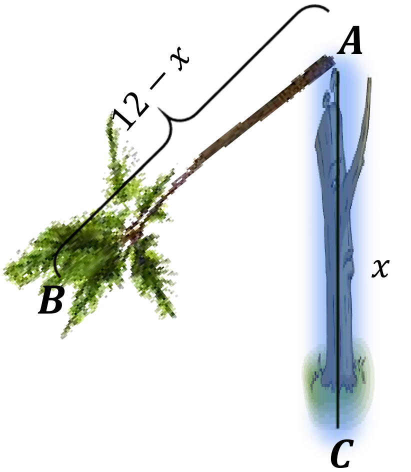
$$x = 216 - 108x + 18x^2 - x^3$$

$$x - 216 + 108x - 18x^2 + x^3 = 0$$

$$x^3 - 18x^2 + 109x - 216 = 0$$

12. A 12 metre tall tree was broken into two parts. It was found that the height of the part which was left standing was the cube root of the length of the part that was cut away. Formulate this into a mathematical problem to find the height of the part which was cut away.

Let x be the height of the part



$$x = \sqrt[3]{12 - x}$$

$$x^3 = 12 - x$$

$$x^3 + x - 12 = 0$$

Exercise 3.2

Example 3.9 Find a polynomial equation of minimum degree with rational coefficients, having $2-\sqrt{3}$ as a root

If one of its root is $2-\sqrt{3}$ *. so other root is* $2+\sqrt{3}$

$$\text{Sum of the root } = 2-\sqrt{3}+2+\sqrt{3}$$

$$\text{Sum of the root} = 4$$

$$\text{Product of the roots} = (2-\sqrt{3})(2+\sqrt{3})$$

$$= (2)^2-(\sqrt{3})^2 = 4-3 = 1$$

$$\text{Product of the roots} = 1$$

$$x^2 - (\text{sum of the root})x + \text{product of the root} = 0$$

$$x^2 - 4x + 1 = 0$$

Example 3.9 Find a polynomial equation with integer coefficients, $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as a root

Since $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ *is a root of the equation. Hence* $x-\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ *is a factors*

To remove the outermost square root. we take $x+\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ *as another factor*

The product of their factors $\left[x-\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right]\left[x+\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right] = 0 \Rightarrow x^2-\frac{\sqrt{2}}{\sqrt{3}} = 0$

Still we didn't achieve our goal. So we include another factor $x^2+\frac{\sqrt{2}}{\sqrt{3}}$

Hence the product is $\left[x^2-\frac{\sqrt{2}}{\sqrt{3}}\right]\left[x^2+\frac{\sqrt{2}}{\sqrt{3}}\right] = 0 \Rightarrow (x^2)^2-\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2 = 0$

$$x^4-\frac{2}{3} = 0 \Rightarrow \frac{3x^4-2}{3} = 0$$

$$3x^4-2 = 0$$

is a required polynomial equation with the integer coefficients.

***Eg*: 3.11 Show that the equation $2x^2 - 6x + 7 = 0$ cannot be satisfied by any real values of x.**

Equation: $2x^2 - 6x + 7 = 0$

$a = 2, b = -6, c = 7$

$\Delta = b^2 - 4ac$

$\Delta = (-6)^2 - 4(2)(7)$

$= 36 - 56 = -20 < 0$

$\therefore \Delta < 0$, *The roots are imaginary numbers*

***Example 3.12*: If $x^2 + 2(k+2)x + 9k = 0$ has equal roots, find K.**

Equation : $x^2 + 2(k+2)x + 9k = 0$

$a = 1, b = 2(k+2), c = 9k$

Given equation has equal roots $\therefore \Delta = 0$

$b^2 - 4ac = 0$

$[2(k+2)]^2 - 4(1)(9k) = 0 \Rightarrow 4(k+2)^2 - 36k = 0$

$4[k^2 + 2^2 + 2(k)(2)] - 36k = 0 \Rightarrow 4[k^2 + 4 + 4k] - 36k = 0$

$4k^2 + 16 + 16k - 36k = 0 \Rightarrow 4k^2 - 20k + 16 = 0$

$\div 4$

$k^2 - 5k + 4 = 0$

$(k-1)(k-4) = 0 \Rightarrow k = 1 = 0, k - 4 = 0$

$k = 1, k = 4$

***Example 3.13*: Show that if p, q, r are rational the roots of the equation $x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$ are rational**

Equation : $x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$

$a = 1, b = -2p, c = p^2 - q^2 + 2qr - r^2$

$\Delta = b^2 - 4ac$

$= (-2p)^2 - 4(1)(p^2 - q^2 + 2qr - r^2)$

$= 4p^2 - 4(p^2 - q^2 + 2qr - r^2)$

$= 4p^2 - 4p^2 + 4q^2 - 8qr + 4r^2 = 4q^2 - 8qr + 4r^2$

$= 4(q^2 - 2qr + r^2) = 4(q-r)^2 > 0$ *which is a perfect equare.*

Hence the roots are rational.

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***Example 3.15**: If $2+i$ and $3-\sqrt{2}$ are roots of the equation*
$\boldsymbol{x^6-13x^5+62x^4-126x^3+65x^2+127x-140=0}$. ***find all roots***

Given roots are $2+i$ *and* $3-\sqrt{2}$,

$2-i$ *and* $3+\sqrt{2}$ *are also roots of the equation*

Hence the factors are

$[x-(2+i)],[x-(2-i)],\left[x-\left(3-\sqrt{2}\right)\right],\left[x-\left(3+\sqrt{2}\right)\right]$ *are factors*

The product of their factors

$$[x-(2+i)][x-(2-i)]\left[x-\left(3-\sqrt{2}\right)\right]\left[x-\left(3+\sqrt{2}\right)\right]=0$$

$$[x^2-(2-i)x-(2+i)x+(2+i)(2-i)]$$

$$\left[x^2-\left(3+\sqrt{2}\right)x-\left(3-\sqrt{2}\right)x+\left(3-\sqrt{2}\right)\left(3+\sqrt{2}\right)\right]=0$$

$$[x^2-2x+\cancel{ix}-2x-\cancel{ix}+2^2+1^2]$$

$$\left[x^2-3x-\cancel{\sqrt{2}x}-3x+\cancel{\sqrt{2}x}+3^2-\left(\sqrt{2}\right)^2\right]=0$$

$$(x^2-4x+5)(x^2-6x+9-2)=0$$

$$(x^2-4x+5)(x^2-6x+7)=0$$

$$x^4-6x^3+7x^2-4x^3+24x^2-28x+5x^2-30x+35=0$$

$$x^4-10x^3+36x^2-58x+35=0$$

Dividing the given polynomial by this factor

$$\begin{array}{r|l}
 & x^2-3x-4 \\
\hline
x^4-10x^3+36x^2-58x+35 & \cancel{x^6}-13x^5+62x^4-126x^3+65x^2+127x-140 \\
 & (-)\quad(+)\quad(-)\quad(+)\quad(-) \\
 & \cancel{x^6}-10x^5+36x^4-58x^3+35x^2 \\
\hline
 & -\cancel{3x^5}+26x^4-68x^3+30x^2+127x \\
 & (+)\quad(-)\quad(+)\quad(-)\quad(+) \\
 & -\cancel{3x^5}+30x^4-108x^3+174x^2-105x \\
\hline
 & -4x^4+40x^3-144x^2+232x-140 \\
 & (+)\quad(-)\quad(+)\quad(-)\quad(+) \\
 & -4x^4+40x^3-144x^2+232x-140 \\
\hline
 & 0
\end{array}$$

The other factor is $x^2-3x-4=0$

$$(x+1)(x-4)=0$$

$$x+1=0,x-4=0$$

$$x=-1,x=4$$

The roots of given polynomial equation are $2+i,2-i,3+\sqrt{2},-1$ *and* 4

1. ***If k is real, discuss the nature of the roots of the polynomial equation $2x^2 + kx + k = 0$ in terms of k.***

$$2x^2 + kx + k = 0$$
$$a = 2, b = k, c = k$$
$$\Delta = b^2 - 4ac$$
$$= k^2 - 4(2)(k)$$
$$\Delta = k^2 - 8k \implies \Delta = k(k-8)$$

When $k < 0, (\Delta > 0)$ *the polynomial has real roots*

When $k = 0, k = 8, (\Delta = 0)$ *the polynomial has equal*

When $0 < k < 8, (\Delta < 0)$ *the polynomial has imaginary*

When $k > 8, (\Delta > 0)$ *the polynomial has real and distinct.*

2. ***Find a polynomial equation of minimum degree with rational coefficients, having $2 + \sqrt{3}\,i$ as a root***

If one of its root is $2 + \sqrt{3}\,i$ *. So the other root is* $2 - \sqrt{3}\,i$

Sum of the roots $= 2 + \sqrt{3}\,i + 2 - \sqrt{3}\,i$

Sum of the roots $= 4$

Product of the roots $= (2 + \sqrt{3}\,i)(2 - \sqrt{3}\,i)$

$$= 2^2 + (\sqrt{3})^2 = 4 + 3 = 7$$

Product of the roots $= 7$

Required equation: $x^2 - (\textit{sum of the root})x + \textit{product of the roots} = 0$

$$x^2 - 4x + 7 = 0$$

3. ***Find a polynomial equation of minimum degree with rational coefficients, having $2i + 3$ as a root***

If one of its root is $3 + 2i$*. So the other roots is* $3 - 2i$

Sum of the roots $= 3 + 2i + 3 - 2i$

Sum of the roots $= 6$

Product of the roots $= (3 + 2i)(3 - 2i) = 3^2 + 2^2 = 9 + 4$

Product of the roots $= 13$

Required equation:

$$x^2 - (\textit{sum of the root})x + \textit{product of the roots} = 0$$
$$x^2 - 6x + 13 = 0$$

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4. Find a polynomial equation of minimum degree with rational coefficients, having $\sqrt{5}-\sqrt{3}$ as a root

If one of its root is $\sqrt{5}-\sqrt{3}$ *. so other root is* $\sqrt{5}+\sqrt{3}$

$$\text{Sum of the root} = \sqrt{5} - \sqrt{3} + \sqrt{5} + \sqrt{3}$$

$$\text{Sum of the root} = 2\sqrt{5}$$

$$\text{Product of the roots} = \left(\sqrt{5} - \sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)$$

$$= \left(\sqrt{5}\right)^2 - \left(\sqrt{3}\right)^2 = 5 - 3 = 2$$

$$\text{Product of the roots} = 2$$

$$x^2 - (\text{sum of the root})x + \text{product of the root} = 0$$

$$x^2 - 2\sqrt{5}\,x + 2 = 0 \Rightarrow x^2 + 2 = 2\sqrt{5}x$$

$$(x^2+2)^2 = \left(2\sqrt{5}\,x\right)^2$$

$$x^4 + 2x^2(2) + 2^2 = 4(5)x^2$$

$$x^4 + 4x^2 + 4 = 20x^2 \Rightarrow x^4 + 4x^2 + 4 - 20x^2 = 0$$

$$x^4 - 16x^2 + 4 = 0$$

Exercise 3.3

Example 3.16: Solve the equation: $\boldsymbol{x^4 - 9x^2 + 20 = 0}$

$x^4 - 9x^2 + 20 = 0$

$(x^2)^2 - 9x^2 + 20 = 0$

$Let\ y = x^2$

$y^2 - 9y + 20 = 0 \Longrightarrow (y-4)(y-5) = 0$

$y - 4 = 0,\ y - 5 = 0 \Longrightarrow y = 4,\ y = 5$

$Now\ x^2 = 4,\ x^2 = 5 \Longrightarrow x = \sqrt{4},\ x = \pm\sqrt{5}$

$x = \pm 2$

The solution of given equation $2, -2, \sqrt{5}, -\sqrt{5}$

$+$ -9 | $\times$ 20 → -4, -5

Example 3.18: Solve the equation: $\boldsymbol{2x^3 + 11x^2 - 9x - 18 = 0}$

Sum of the coefficient of odd terms

$= Sum\ of\ the\ coefficient\ of\ even\ terms$

-1 *is a root of the equation and Hence* $x + 1$ *is a factor of the equation*

Divide $2x^3 + 11x^2 - 9x - 18$ *by* $x + 1$

$$\begin{array}{r|l} & 2x^2 + 9x - 18 \\ \hline x+1 & 2x^3 + 11x^2 - 9x - 18 \\ & (-)\quad (-) \\ & 2x^3 + 2x^2 \\ \hline & \quad 9x^2 - 9x \\ & (-)\quad (-) \\ & \quad 9x^2 + 9x \\ \hline & \quad -18x - 18 \\ & \quad (+)\quad (+) \\ & \quad -18x - 18 \\ \hline & \quad 0 \end{array}$$

$\therefore$ *The other factor is* $2x^2 + 9x - 18 = 0$

$2x^2 + 9x - 18 = 0$

$(x+6)(2x-3) = 0 \Longrightarrow x + 6 = 0, 2x - 3 = 0$

$x = -6,\ 2x = 3$

$x = \dfrac{3}{2}$

$+$ 9 | $\times$ -36 → 6 ($\frac{12x}{2x^2} = \frac{6}{x}$), -3 ($\frac{-3x}{2x^2}$ → x)

$\therefore$ *The roots of the given equation are* $-1, -6, \dfrac{3}{2}$

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Example 3.22: It is known that the roots of the equation $x^3-6x^2-4x+24=0$ are in arithmetic progression. Find its roots

Let α,β,γ be the roots of the equation form an A.P.

$\alpha=a-d, \beta=a, \gamma=a+d$

Given equation : $x^3-6x^2-4x+24=0$

$a=1, b=-6, c=-4, d=24$

sum of the roots: $\Sigma_1=-\frac{b}{a}=\frac{6}{1}=6$

$\alpha+\beta+\gamma=6$

$a-d+a+a+d=6 \Rightarrow 3a=6$

$a=2$

Hence $\beta=2$

Product of the roots: $\Sigma_3=-\frac{d}{a}$

$\Sigma_3=\frac{-24}{1}=-24 \Rightarrow \alpha\beta\gamma=-24$

$\alpha\times2\times\gamma=-24 \Rightarrow \alpha\gamma=-12$

$(a-d)(a+d)=-12 \Rightarrow a^2-d^2=-12$ where $a=2$

$2^2-d^2=-12 \Rightarrow 4-d^2=-12 \Rightarrow -d^2=-12-4$

$-d^2=-16 \Rightarrow d^2=16 \Rightarrow d=\sqrt{16}$

$d=\pm4$

$\alpha=a-d=2-4=-2$

$\gamma=a+d=2+4=6$

The roots of the equation are $-2, 2, 6$

1. Solve the cubic equation : $2x^3-x^2-18x+9=0$ if sum of two of its roots vanishes.

Given equation : $2x^3-x^2-18x+9=0$

$a=2, b=-1, c=-18, d=9$

Let α,β,γ be the roots of the equation.

Given sum of two of its roots vanishes : $\alpha+\beta=0$

$\beta=-\alpha$

Sum of the roots : $\sum_1\alpha=\alpha+\beta+\gamma=-\frac{b}{a} \Rightarrow \alpha+\beta+\gamma=\frac{1}{2}$

where $\beta=-\alpha$

$\alpha-\alpha+\gamma=\frac{1}{2} \Rightarrow \gamma=\frac{1}{2}$

Product of the roots: $\sum_3\alpha\beta\gamma=-\frac{d}{a}$

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$$\alpha\beta\gamma = -\frac{9}{2} \quad where\ \beta = -\alpha\ and\ \gamma = \frac{1}{2}$$

$$\alpha \times -\alpha \times \frac{1}{2} = -\frac{9}{2} \Rightarrow \alpha^2 \times \frac{1}{2} = \frac{9}{2}$$

$$\alpha^2 = 9 \Rightarrow \alpha = \sqrt{9} \Rightarrow \alpha = \pm 3$$

$$sub\ \alpha = 3\ in\ \beta = -\alpha \Rightarrow \beta = -3$$

$$\therefore \alpha = 3, \beta = -3, \gamma = 1/2$$

4. Determine k and solve the equation $2x^3 - 6x^2 + 3x + k = 0$ if one of its roots is twice the sum of the other two roots.

$$Equation: 2x^3 - 6x^2 + 3x + k = 0$$

$$a = 2, b = -6, c = 3, d = k$$

$$let\ \alpha, \beta, \gamma\ are\ the\ roots\ of\ the\ equation$$

$$Given: one\ roots = 2 \times sum\ of\ the\ other\ two\ roots$$

$$\alpha = 2(\beta + \gamma)$$

$$Sum\ of\ the\ roots: \sum_1 \alpha = -\frac{b}{a}$$

$$\sum_1 = \frac{6}{2} = 3 \Rightarrow \sum_1 = 3$$

$$\alpha + \beta + \gamma = 3 \ldots (1)$$

$$2(\beta + \gamma) + \beta + \gamma = 3 \Rightarrow 3(\beta + \gamma) = 3$$

$$\beta + \gamma = 1 \quad \ldots (2)$$

$$Sub\ \beta + \gamma = 1\ in\ (1)\ \alpha + \beta + \gamma = 3$$

$$\alpha + 1 = 3 \Rightarrow \alpha = 3 - 1$$

$$\boxed{\alpha = 2}$$

$$The\ sum\ of\ product\ of\ two\ roots: \sum_2 = \frac{c}{a} = \frac{3}{2}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{3}{2} \quad where\ \alpha = 2$$

$$2\beta + \beta\gamma + 2\gamma = \frac{3}{2} \Rightarrow 2\beta + 2\gamma + \beta\gamma = \frac{3}{2}$$

$$2(\beta + \gamma) + \beta\gamma = \frac{3}{2} \quad where\ \beta + \gamma = 1$$

$$2(1) + \beta\gamma = \frac{3}{2} \Rightarrow \beta\gamma = \frac{3}{2} - 2 \Rightarrow \beta\gamma = \frac{3-4}{2} \Rightarrow \beta\gamma = -\frac{1}{2}$$

$$The\ sum\ of\ product\ of\ all\ roots: \sum_3 = -\frac{d}{a} = -\frac{k}{2}$$

$$\alpha\beta\gamma = -\frac{k}{2}$$

$$\alpha\beta\gamma = -\frac{k}{2} \quad where\ \alpha = 2, \beta\gamma = -\frac{1}{2}$$

$$2\left(\frac{-1}{2}\right) = -\frac{k}{2} \Rightarrow -1 = \frac{-k}{2} \quad \therefore k = 2$$

$$\beta\gamma = -\frac{1}{2} \qquad \beta + \gamma = 1$$

$$\beta = 1 - \gamma$$

$$(1-\gamma)\gamma = -\frac{1}{2} \Rightarrow \gamma - \gamma^2 = -\frac{1}{2}$$

$$2\gamma - 2\gamma^2 = -1 \Rightarrow 2\gamma - 2\gamma^2 + 1 = 0$$

$$-2\gamma^2 + 2\gamma + 1 = 0 \Rightarrow 2\gamma^2 - 2\gamma - 1 = 0$$

$$a = 2, b = -2, c = -1$$

$$\gamma = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \gamma = \frac{2 \pm \sqrt{4 - 4(2)(-1)}}{2(2)}$$

$$\gamma = \frac{2 \pm \sqrt{4+8}}{4} \Rightarrow \gamma = \frac{2 \pm \sqrt{12}}{4} \Rightarrow \gamma = \frac{2 \pm \sqrt{4 \times 3}}{4}$$

$$\gamma = \frac{2 \pm 2\sqrt{3}}{4} \Rightarrow \gamma = \frac{2(1 \pm \sqrt{3})}{4\ 2} \Rightarrow \gamma = \frac{1 \pm \sqrt{3}}{2}$$

$$\beta = 1 - \gamma \Rightarrow \beta = 1 - \frac{1 \pm \sqrt{3}}{2} = \frac{2 - (1 \pm \sqrt{3})}{2} = \frac{2 - 1 \pm \sqrt{3}}{2} = \frac{1 \pm \sqrt{3}}{2}$$

$$\beta = \frac{1 \pm \sqrt{3}}{2}$$

$$\therefore \textit{The roots are } \alpha = 2, \beta = \frac{1 \pm \sqrt{3}}{2}, \gamma = \frac{1 \pm \sqrt{3}}{2}$$

5. Find all zeros of the polynomial $x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135$ if it is known that $1 + 2i$ and $\sqrt{3}$ are two of its zeros.

Given zeros of the polynomial are $1 + 2i$ *and* $\sqrt{3}$

$1 - 2i$ *and* $-\sqrt{3}$ *are also zeros of the polynomial.*

The factors are $[x - (1+2i)], [x - (1-2i)], (x - \sqrt{3}), (x + \sqrt{3})$

Hence the product of the factors

$$[x - (1+2i)][x - (1-2i)](x - \sqrt{3})(x + \sqrt{3})$$

$$[\underbrace{x-1}_{a} \underbrace{-2i}_{b}][\underbrace{x-1}_{a} \underbrace{+2i}_{b}](x - \sqrt{3})(x + \sqrt{3})$$

$$[(x-1)^2 + (2)^2]\left[x^2 - \left(\sqrt{3}\right)^2\right] = (x^2 - 2x + 1 + 4)(x^2 - 3)$$

$$= (x^2 - 2x + 5)(x^2 - 3) = x^4 - 2x^3 + 5x^2 - 3x^2 + 6x - 15$$

$$= x^4 - 2x^3 + 2x^2 + 6x - 15$$

Dividing the given polynomial by this factor: $x^4 - 2x^3 + 2x^2 + 6x - 15$

$$
\begin{array}{r|l}
 & x^2 - x - 9 \\
\hline
x^4 - 2x^3 + 2x^2 + 6x - 15 & x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135 \\
 & \underset{(-)}{x^6} \underset{(+)}{-2x^5} \underset{(-)}{+2x^4} \underset{(-)}{+6x^3} \underset{(+)}{-15x^2} \\
\hline
 & -x^5 - 7x^4 + 16x^3 - 24x^2 - 39x \\
 & \underset{(+)}{-x^5} \underset{(-)}{+2x^4} \underset{(+)}{-2x^3} \underset{(+)}{-6x^2} \underset{(-)}{+15x} \\
\hline
 & -9x^4 + 18x^3 - 18x^2 - 54x + 135 \\
 & \underset{(+)}{-9x^4} \underset{(-)}{+18x^3} \underset{(+)}{-18x^2} \underset{(+)}{-54x} \underset{(-)}{+135} \\
\hline
 & 0
\end{array}
$$

$\therefore$ *The other factor is* $x^2 - x - 9 = 0$

$a = 1, b = -1, c = -9$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \implies x = \frac{1 \pm \sqrt{1^2 - 4(1)(-9)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{1 + 36}}{2} = \frac{1 \pm \sqrt{37}}{2}$$

$$x = \frac{1 + \sqrt{37}}{2}, \frac{1 - \sqrt{37}}{2}$$

6. *Solve*: (*i*) $\mathbf{2x^3 - 9x^2 + 10x - 3 = 0}$

Sum of the coefficient $= 2 - 9 + 10 - 3 = 12 - 12 = 0$

$\therefore (x - 1)$ *is the one of the factor of the given polynomial*

Divide: $2x^3 - 9x^2 + 10x - 3$ *by* $x - 1$

$$
\begin{array}{r|l}
 & 2x^2 - 7x + 3 \\
\hline
x - 1 & 2x^3 - 9x^2 + 10x - 3 \\
 & \underset{(-)}{2x^3} \underset{(+)}{-2x^2} \\
\hline
 & -7x^2 + 10x \\
 & \underset{(+)}{-7x^2} \underset{(-)}{+7x} \\
\hline
 & 3x - 3 \\
 & \underset{(-)}{3x} \underset{(+)}{-3} \\
\hline
 & 0
\end{array}
$$

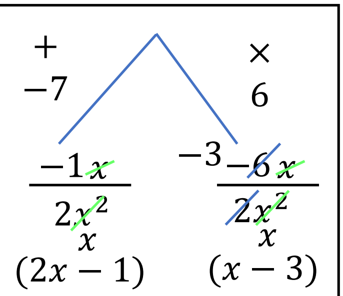
we get the other factor is $2x^2 - 7x + 3 = 0$

$(2x-1)(x-3)=0 \implies 2x-1=0, \quad x-3=0$

$2x=1, x=3$

$x=\frac{1}{2}$

$\therefore$ *The roots of the given polynomial are* $1, \frac{1}{2}, 3$



6. (ii) $\mathbf{8x^3 - 2x^2 - 7x + 3 = 0}$

Sum of the co.efficient of odd terms $= 8 - 7 = 1$

Sum of the co.efficient of even terms $= -2 + 3 = 1$

Sum of the co.efficient of odd terms

$=$ *Sum of the co.efficient of even terms*

$\therefore (x+1)$ *is one of the factor of the given polynomial.*

hence $x=-1$ *is one of the root*

Divide: $8x^2 - 2x^2 - 7x + 3$ *by* $x+1$

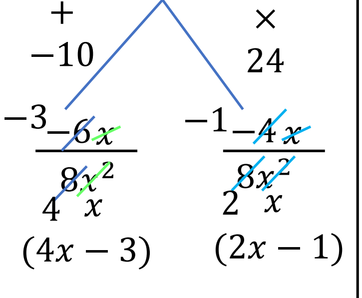
$$
\begin{array}{r|l}
 & 8x^2 - 10x + 3 \\
\hline
x+1 & 8x^3 - 2x^2 - 7x + 3 \\
 & (-)\quad(-) \\
 & 8x^3 + 8x^2 \\
\hline
 & -10x^2 - 7x \\
 & (+)\quad(+) \\
 & -10x^2 - 10x \\
\hline
 & 3x + 3 \\
 & (-)\quad(-) \\
 & 3x + 3 \\
\hline
 & 0
\end{array}
$$

we get the other factor is $8x^2 - 10x + 3 = 0$

$(4x-3)(2x-1)=0 \implies 4x-3=0, 2x-1=0$

$4x=3, \ 2x=1$

$x=\frac{3}{4}, x=\frac{1}{2}$



$\therefore$ *The roots of the given polynomial are* $-1, \frac{1}{2}, \frac{3}{4}$

7. *Solve the equation*: $\mathbf{x^4 - 14x^2 + 45 = 0}$

$x^4 - 14x^2 + 45 = 0 \implies (x^2)^2 - 14x^2 + 45 = 0$

Let $y = x^2$

$y^2 - 14y + 45 = 0 \implies (y-9)(y-5)=0 \implies y-9=0, \ y-5=0$

$y=9, \ y=5$

Now $x^2=9, \ x^2=5 \implies x=\sqrt{9}, \ x=\pm\sqrt{5}$

$x=\pm 3$

The solution of a given equation $3, -3, \sqrt{5}, -\sqrt{5}$

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Exercise: 3.5

Example 3.25: Solve the equation: $x^3 - 5x^2 - 4x + 20 = 0$

Here 2 is a root of the equation. Hence $x - 2$ is a factor of the polynomial

Divide $x^3 - 5x^2 - 4x + 20$ by $x - 2$

Quotient: $x^2 - 3x - 10$

$x - 2 \,)\; x^3 - 5x^2 - 4x + 20$

$(-)\;(+)$

$x^3 - 2x^2$

$-3x^2 - 4x$

$(+)\;(-)$

$-3x^2 + 6x$

$-10x + 20$

$(+)\;(-)$

$-10x + 20$

0

$\therefore$ *The other factor is* $x^2 - 3x - 10 = 0$

$(x-5)(x+2) = 0$

$x - 5 = 0,\ x + 2 = 0$

$x = 5, \quad x = -2$

The solutions are $2, -2, 5$

$+$: -3, $\times$: -10; -5, 2

Rational theorem:

Let $a_n x^n + \cdots + a_1 x + a_0$ *with* $a_n \neq 0\ a_0 \neq 0$. *If* $\frac{p}{q}$ *with* $G.c.d\ (p,q) = 1$

$$\frac{p}{q} = \frac{\pm\, factor\ of\ a_0}{\pm\, factor\ of\ a_n}$$

Example 3.26: Find the roots of $2x^3 + 3x^2 + 2x + 3$

$a_n = 2 : \pm(1,2)$

$a_0 = 3 : \pm(1,3)$

$$\frac{p}{q} = \frac{\pm\, factor\ of\ a_0}{\pm\, factor\ of\ a_n}$$

$$\frac{p}{q} = \frac{\pm(1,3)}{\pm(1,2)} = \pm\frac{1}{1}, \pm\frac{3}{1}, \pm\frac{1}{2}, \pm\frac{3}{2}$$

are the possible rational roots of the polynomial

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Rational roots	polynomial $P(x) = 2x^3 + 3x^2 + 2x + 3$	$P(x)$	Root/not a root
1	$2 + 3 + 2 + 3$	$\neq 0$	not a root
-1	$-2 + 3 - 2 + 3$	$\neq 0$	not a root
3	$2(27) + 3(9) + 2(3) + 3$	$\neq 0$	not a root
-3	$2(-27) + 3(9) - 6 + 3$ $= -54 + 27 - 6 + 3$	$\neq 0$	not a root

Rational roots	polynomial $P(x) = 2x^3 + 3x^2 + 2x + 3$	$P(x)$	Root/not a root
$\frac{1}{2}$	$= 2\left(\frac{1}{8}\right) + 3\left(\frac{1}{4}\right) + 2\left(\frac{1}{2}\right) + 3$	$\neq 0$	not a root
$-\frac{1}{2}$	$= 2\left(-\frac{1}{8}\right) + 3\left(\frac{1}{4}\right) + 2\left(-\frac{1}{2}\right) + 3$ $= -\frac{1}{4} + \frac{3}{4} - 1 + 3$	$\neq 0$	not a root
$\frac{3}{2}$	$= 2\left(\frac{3}{2}\right)^3 + 3\left(\frac{3}{2}\right)^2 + 2\left(\frac{3}{2}\right) + 3$	$\neq 0$	not a root
$-\frac{3}{2}$	$= 2\left(-\frac{3}{2}\right)^3 + 3\left(-\frac{3}{2}\right)^2 + 2\left(-\frac{3}{2}\right) + 3$ $= 2\left(-\frac{27}{8}\right) + 3\left(\frac{9}{4}\right) - 3 + 3$	$= 0$	is a root

$\therefore -\frac{3}{2}$ *is a rational root*

$x = -\frac{3}{2} \Rightarrow 2x = -3$

$2x + 3 = 0$ *is a factor of the given polynomial*

To find other factor Divide: $2x^3 + 3x^2 + 2x + 3$ by $2x + 3$

$$
\begin{array}{r|l}
 & x^2 + 1 \\
\hline
2x + 3 & 2x^3 + 3x^2 + 2x + 3 \\
 & (-)\quad(-) \\
 & 2x^3 + 3x^2 \\
\hline
 & 2x + 3 \\
 & (-)\quad(-) \\
 & 2x + 3 \\
\hline
 & 0
\end{array}
$$

Example 3.27: Solve the equation $7x^3 - 43x^2 = 43x - 7$

$Equation: 7x^3 - 43x^2 - 43x + 7 = 0$

This is an odd degree reciprocal equation of Type I.

-1 *is a root of the equation and have* $x + 1$ *is a factor.*

Dividing the polynomial $7x^3 - 43x^2 - 43x + 7$ *by the factor* $x + 1$

$$\begin{array}{r|l} & 7x^2 - 50x + 7 \\ \hline x+1 & 7x^3 - 43x^2 - 43x + 7 \\ & \;(-)\quad (-) \\ & 7x^3 + 7x^2 \\ \hline & -50x^2 - 43x \\ & \;(+)\quad (+) \\ & -50x^2 - 50x \\ \hline & 7x + 7 \\ & (-)\ (-) \\ & 7x + 7 \\ \hline & 0 \end{array}$$

$+$ -50 $\times$ 49 -7

$\frac{-1x}{7x^2}\,x$ $\frac{-49x}{7x^2}\,x$

$(7x - 1)$ $(x - 7)$

$\therefore$ *The other factor is* $7x^2 - 50x + 7 = 0$

$$(7x - 1)(x - 7) = 0 \Rightarrow 7x - 1 = 0, x - 7 = 0$$

$$7x = 1, x = 7$$

$$x = \frac{1}{7}$$

Solution of the given equations are $-1, \frac{1}{7}, 7.$

Example 3.28: Solve the equation: $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$

$$x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$$

$$\div x^2$$

$$x^2 - 10x + 26 - \frac{10}{x} + \frac{1}{x^2} = 0 \Rightarrow x^2 + \frac{1}{x^2} - 10x - \frac{10}{x} + 26 = 0$$

$$x^2 + \frac{1}{x^2} - 10\left(x + \frac{1}{x}\right) + 26 = 0$$

Let $y = x + \frac{1}{x}$ *then* $x^2 + \frac{1}{x^2} = y^2 - 2$

$$y^2 = \left(x + \frac{1}{x}\right)^2 \Rightarrow y^2 = x^2 + \frac{1}{x^2} + 2(x)\left(\frac{1}{x}\right)$$

$$y^2 = x^2 + \frac{1}{x^2} + 2 \Rightarrow y^2 - 2 = x^2 + \frac{1}{x^2}$$

$$y^2 - 2 - 10y + 26 = 0 \Rightarrow y^2 - 10y + 24 = 0$$

$$(y - 6)(y - 4) = 0 \Rightarrow y - 6 = 0, y - 4 = 0$$

$$y = 6, y = 4$$

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$Case(i)\ y = 6$

$$x + \frac{1}{x} = 6 \implies \frac{x^2+1}{x} = 6 \implies x^2 + 1 = 6x$$

$$x^2 - 6x + 1 = 0$$

$$a = 1, b = -6, c = 1$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \implies x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{6 \pm \sqrt{32}}{2} \implies x = \frac{6 \pm \sqrt{16 \times 2}}{2}$$

$$x = \frac{6 \pm 4\sqrt{2}}{2} \implies x = 3 \pm 2\sqrt{2}$$

$$x = 3 + 2\sqrt{2}, 3 - 2\sqrt{2}$$

$Case\ (ii)\ y = 4$

$$x + \frac{1}{x} = 4 \implies \frac{x^2+1}{x} = 4 \implies x^2 + 1 = 4x$$

$$x^2 - 4x + 1 = 0$$

$$a = 1, b = -4, c = 1$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \implies x = \frac{4 \pm \sqrt{16 - 4(1)(1)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2}$$

$$x = 2 \pm \sqrt{3} \implies x = 2 + \sqrt{3}, 2 - \sqrt{3}$$

3. *Solve*: $8x^{\frac{3}{2n}} - 8x^{-\frac{3}{2n}} = 63$

$$8x^{\frac{3}{2n}} - \frac{8}{x^{\frac{3}{2n}}} = 63 \implies 8\left(x^{\frac{3}{2n}} - \frac{1}{x^{\frac{3}{2n}}}\right) = 63$$

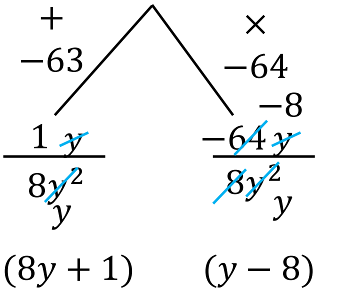
$$Let\ y = x^{\frac{3}{2n}}$$

$$8\left(y - \frac{1}{y}\right) = 63 \implies \frac{8(y^2-1)}{y} = 63$$

$$8y^2 - 8 = 63y \implies 8y^2 - 63y - 8 = 0$$

$$(8y+1)(y-8) = 0 \implies 8y = -1, y = 8$$

$$y = -\frac{1}{8},$$



Now $y = x^{\frac{3}{2n}}$

$$x^{\frac{3}{2n}} = -\frac{1}{8}, x^{\frac{3}{2n}} = 8$$

$$x^{\frac{1}{2n}} = \left(-\frac{1}{8}\right)^{1/3},\ x^{\frac{1}{2n}} = (8)^{1/3} \implies x^{\frac{1}{2n}} = -\frac{1}{2}, x^{\frac{1}{2n}} = 2$$

$$x = \left(-\frac{1}{2}\right)^{2n},\ x = (2)^{2n} \implies x = \left(\frac{1}{4}\right)^{n}, x = 4^n$$

$$\boxed{x = 4^{-n}}$$

4. Solve: $\boldsymbol{2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}}$

$$2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b} \implies 2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b^2+6a^2}{ab}$$

$$2\sqrt{\frac{x}{a}} + \frac{3}{\sqrt{\frac{x}{a}}} = \frac{b^2+6a^2}{ab}$$

$$Let\ y = \sqrt{\frac{x}{a}}$$

$$2y + \frac{3}{y} = \frac{b^2+6a^2}{ab} \implies \frac{2y^2+3}{y} = \frac{b^2+6a^2}{ab}$$

$$2aby^2 + 3ab = b^2y + 6a^2y \implies 2aby^2 - b^2y - 6a^2y + 3ab = 0$$

$$2aby^2 - b^2y - 6a^2y + 3ab = 0 \implies by(2ay - b) - 3a(2ay - b) = 0$$

$$(2ay - b)(by - 3a) = 0 \implies 2ay - b = 0,\ by - 3a = 0$$

$$2ay = b,\ by = 3a \implies y = \frac{b}{2a},\ \ y = \frac{3a}{b}$$

$$Case(i)\ \ y = \frac{b}{2a}$$

$$\sqrt{\frac{x}{a}} = \frac{b}{2a} \implies \frac{x}{a} = \frac{b^2}{4a^2} \implies x = \frac{b^2}{4a}$$

squaring on both side

$$Case\ (ii)\ y = \frac{3a}{b}$$

$$\sqrt{\frac{x}{a}} = \frac{3a}{b} \implies \frac{x}{a} = \frac{9a^2}{b^2} \implies x = \frac{9a^3}{b^2}$$

Squaring on both side

5. Solve the equations (i) $\boldsymbol{6x^6 - 35x^5 + 56x^4 - 56x^2 + 35x - 6 = 0}$

The polynomial equation : $6x^6 - 35x^5 + 56x^4 - 56x^2 + 35x - 6 = 0$

(*Middle is zero*)

Which is an even degree reciprocal equation of Type II.

So 1 *and* -1 *are two solution.*

Hence $(x-1)$ *and* $(x+1)$ *is a factor of the polynomial.*

The product of the factor $=(x-1)(x+1)=x^2-1$

Dividing the polynomial by the factor x^2-1

$$
\begin{array}{r|l}
 & 6x^4-35x^3+62x^2-35x+6 \\
\hline
x^2-1 & 6x^6-35x^5+56x^4-56x^2+35x-6 \\
 & (-)\quad(+) \\
 & 6x^6-6x^4 \\
\hline
 & -35x^5+62x^4 \\
 & (+)\quad(-) \\
 & -35x^5+35x^3 \\
\hline
 & 62x^4-35x^3-56x^2 \\
 & (-)\quad(+) \\
 & 62x^4-62x^2 \\
\hline
 & -35x^3+6x^2+35x \\
 & (+)\quad(-) \\
 & -35x^3+35x \\
\hline
 & 6x^2-6 \\
 & (-)\quad(+) \\
 & 6x^2-6 \\
\hline
 & 0
\end{array}
$$

5. *Solve the equations* (*i*) $\mathbf{6x^4-35x^3+62x^2-35x+6=0}$

$6x^4-35x^3+62x^2-35x+6$ *as a factor Dividing this factor by* x^2 *and rearranging the terms we get*

$$\frac{6x^4-35x^3+62x^2-35x+6}{x^2}=0$$

$$6x^2-35x+62-\frac{35}{x}+\frac{6}{x^2}=0 \Rightarrow 6x^2+\frac{6}{x^2}-35x-\frac{35}{x}+62=0$$

$$6\left(x^2+\frac{1}{x^2}\right)-35\left(x+\frac{1}{x}\right)+62=0$$

Let $y=x+\dfrac{1}{x}$

$$y^2=\left(x+\frac{1}{x}\right)^2 \Rightarrow y^2=x^2+\frac{1}{x^2}+2(x)\left(\frac{1}{x}\right)$$

$$y^2=x^2+\frac{1}{x^2}+2 \Rightarrow y^2-2=x^2+\frac{1}{x^2}$$

$$6(y^2-2)-35y+62=0 \Rightarrow 6y^2-12-35y+62=0$$

$$6y^2-35y+50=0 \Rightarrow (2y-5)(3y-10)=0$$

+ -35 × 300

-5: $\frac{-15y}{6y^2}$ → $2\ y$ → $(2y-4)$

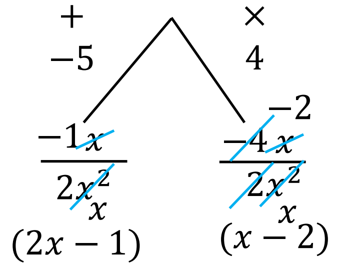
-10: $\frac{-20y}{6y^2}$ → $3\ y$ → $(3y-10)$

$2y = 5, \; 2y = 10$

$y = \frac{5}{2}, \quad y = \frac{10}{3}$

$Take : y = \frac{5}{2}$

$x + \frac{1}{x} = \frac{5}{2} \implies \frac{x^2+1}{x} = \frac{5}{2} \implies 2x^2 + 2 = 5x$

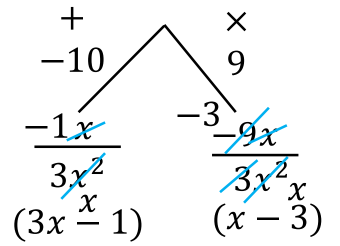


$2x^2 - 5x + 2 = 0 \implies (2x-1)(x-2) = 0 \implies 2x - 1 = 0, \; x - 2 = 0$

$2x = 1, x = 2$

$\boxed{x = \frac{1}{2}}$

$Taking \; y = \frac{10}{3}$



$x + \frac{1}{x} = \frac{10}{3} \implies \frac{x^2+1}{x} = \frac{10}{3}$

$3x^2 + 3 = 10x \implies 3x^2 - 10x + 3 = 0$

$(3x-1)(x-3) = 0 \implies 3x - 1 = 0, x - 3 = 0$

$3x = 1, x = 3$

$\boxed{x = \frac{1}{3}}$

Hence the solutions are $1, -1, 2, \frac{1}{2}, 3, \frac{1}{3}$

5. (ii) $\boldsymbol{x^4 + 3x^3 - 3x - 1 = 0}$

The polynomial equation is an even degree equation of type II and the middle term is zero

$\therefore x = 1$ *and* $x = -1$ *are solution*

Hence the product of the factor $= (x-1)(x+1) = x^2 - 1$

Divide the polynomial by the factor $x^2 - 1$

$$
\begin{array}{r|l}
 & x^2 + 3x + 1 \\
\hline
x^2 - 1 & x^4 + 3x^3 - 3x - 1 \\
 & (-) \quad (+) \\
 & x^4 - x^2 \\
\hline
 & 3x^3 + x^2 - 3x \\
 & (-) \qquad (+) \\
 & 3x^3 - 3x \\
\hline
 & x^2 - 1 \\
 & (-) \quad (+) \\
 & x^2 - 1 \\
\hline
 & 0
\end{array}
$$

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$\therefore$ *The other factor is* $x^2 + 3x + 1 = 0$

$$a = 1, b = 3, c = 1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)} = \frac{-3 \pm \sqrt{9-4}}{2}$$

$$x = \frac{-3 \pm \sqrt{5}}{2} \implies x = \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$$

The solutions are $1, -1, \dfrac{-3+\sqrt{5}}{2}, \dfrac{-3-\sqrt{5}}{2}$

6. *Find all real numbers satisfying* $\mathbf{4^x - 3\left(2^{x+2}\right) + 2^5 = 0}$

$$4^x - 3(2^{x+2}) + 2^5 = 0$$

$$(2^2)^x - 3(2^x)(2^2) + 2^5 = 0$$

$$(2^x)^2 - 12(2^x) + 32 = 0$$

Let $y = 2^x$

$$y^2 - 12y + 32 = 0 \implies (y-8)(y-4) = 0$$

$$y - 8 = 0, y - 4 = 0$$

$$y = 8, y = 4$$

Take: $y = 8$

$$2^x = 8 \implies 2^x = 2^3$$

$$\boxed{x = 3}$$

Take: $y = 4$

$$2^x = 4 \implies 2^x = 2^2$$

$$\boxed{x = 2}$$

7. *Solve the equation*: $\mathbf{6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0}$

if it is known that $\frac{1}{3}$ *is a solution*

Equation: $6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$

Given $\frac{1}{3}$ *is a root of the equation and 3 is also a root of the equation.*

Hence the factors are $(3x-1), (x-3)$

Product of the factors : $(3x-1)(x-3) = 0$

$$3x^2 - 9x \qquad + 3 = 0$$

$$3x^2 - 10x + 3 = 0$$

Dividing the given polynomial by $3x^2 - 10x + 3 = 0$

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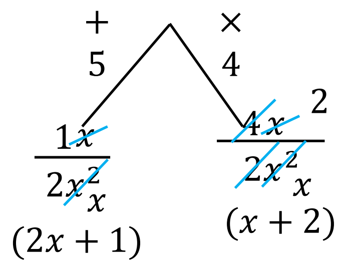
$$
\begin{array}{r|l}
 & 2x^2 + 5x + 2 \\
\hline
3x^2 - 10x + 3 & 6x^4 - 5x^3 - 38x^2 - 5x + 6 \\
 & (-)\quad (+)\quad (-) \\
 & 6x^4 - 20x^3 + 6x^2 \\
\hline
 & 15x^3 - 44x^2 - 5x \\
 & (-)\quad (+)\quad (-) \\
 & 15x^3 - 50x^2 + 15x \\
\hline
 & 6x^2 - 20x + 6 \\
 & (-)\quad (+)\quad (-) \\
 & 6x^2 - 20x + 6 \\
\hline
 & 0
\end{array}
$$

The other factor is $2x^2 + 5x + 2 = 0$



$(2x+1)\,(x+2) = 0$

$2x + 1 = 0,\ x + 2 = 0$

$2x = -1,\ x = -2$

$x = -\frac{1}{2},$

$\therefore$ *The solutions are* $-2, -\frac{1}{2}, 3, \frac{1}{3}$

Exercise: 3.6

Descartes Rule

The number of positive roots, number of negative roots and number of non real complex roots for a polynomial over $\mathbb{R}$

Statement of Descartes Rule

The concept of change of sign in the coefficients of a polynomial

The polynomial : $2x^7 - 3x^6 - 4x^5 + 6x^3 - 7x + 8$

Sign of coefficient : $+, -, -, +, +, -, +$

Definition : 3.2

The number of sign changes, we get some information about the roots of the polynomial using **Descartes Rule**.

Theorem : 3.7

If P is the number of positive zero of a polynomial $p(x)$ with real coefficient and S is the number of sign changes is coefficient of $P(x)$ then $S - P$ is non negative even integer.

The theorem state that the number of positive root of a polynomial $P(x)$ cannot be more than the number of sign changes in coefficient of $P(x)$.

The difference between the number of sign changes in the coefficient of $P(x)$ and the number of positive roots of the polynomial $p(x)$ is even.

Attainments of Bounds

(a) Bounds for the number of real roots

The polynomial : $p(x) = x^5 - 2x^4 - x + 2$

$+, -, -, +$

$1 \quad 0 \quad 1$

$\therefore p(x)$ *has 2 sign changes*

$m = 2$

The number of positive roots of the polynomial 2 or 0

$p(-x) = (-x)^5 - 2(-x)^4 - (-x) + 2$

$p(-x) = -x^5 - 2x^4 + x + 2$

$-, -, +, +$

$0 \quad 1 \quad 0$

$\therefore p(-x)$ *has 1 sign changes*

$k = 1$

The number of negative roots of the polynomial 1

The polynomial $p(x) = x^4 + 5x^3 + 7x^2 + 5x + 6$

$p(x)$ *has no sign changes*

$m = 0$

$p(-x) = x^4 - 5x^3 + 7x^2 - 5x + 6$

$+, -, +, -, +$

1 1 1 1

$p(-x)$ *has 4 sign changes*

By Descrates rule

$p(x)$ *has no positive roots*

$p(-x)$ *has 4 or 2 or 0 negative roots.*

Bounds for the number of Imaginary (Non real complex) roots using the Descartes rule,

To find number of Imaginary roots.

Let m denote the number of sign changes in coefficient of $p(x)$

Let k denote the number of sign changes in coefficient of $p(-x)$

Let $p(x)$ *of degree n*

$\therefore$ *The number of Imaginary roots* $= n - (m + k)$

***Example 3.30*: Show that the polynomial $9x^9 + 2x^5 - x^4 - 7x^2 + 2$**

$p(x) = 9x^9 + 2x^5 - x^4 - 7x^2 + 2$

$+, \quad +, \quad -, \quad +, \quad +$

1 1

$p(-x) = -9x^9 - 2x^5 - x^4 - 7x^2 + 2$

$-, \quad -, \quad -, \quad -, \quad +$

1

Positive roots	*Negative roots*	*Total No. of roots*	*Imaginary roots*
2	1	9	$9 - 3 = 6$
0	1	9	$9 - 1 = 8$

***Example 3.31*: Discuss the nature of the roots of the following polynomial**

(*i*) $x^{2018} + 1947x^{1950} + 15x^8 + 26x^6 + 2019$

$p(x) = x^{2018} + 1947x^{1950} + 15x^8 + 26x^6 + 2019$

$+, \quad +, \quad +, \quad +, \quad +$

$p(-x) = x^{2018} + 1947x^{1950} + 15x^8 + 26x^6 + 2019$

$+, \quad +, \quad +, \quad +, \quad +$

Positive roots (m)	*Negative roots* (k)	*Total No. of roots* n	*Imaginary roots*
0	0	2018	2018

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(ii) $x^5 - 19x^4 + 2x^3 + 5x^2 + 11$

Let $p(x) = x^5 - 19x^4 + 2x^3 + 5x^2 + 11$

$+, \quad -, \quad +, \quad +, \quad +$

1 2

$p(-x) = -x^5 - 19x^4 - 2x^3 + 5x^2 + 11$

$-, \quad -, \quad -, \quad +, \quad +$

1

No. of possibilities

Positive roots M	Negative roots k	Total No. of roots n	Imaginary roots
2	1	5	$5 - 3 = 2$
0	1	5	$5 - 1 = 4$

1. Discuss the Maximum possible number of positive and negative roots of the polynomial equation $9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2 = 0$

Let $p(x) = 9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2$

$+, \quad -, \quad +, \quad -, \quad +, \quad +, \quad +, \quad +, \quad +$

1 2 3 4

$p(-x) = -9x^9 - 4x^8 - 4x^7 - 3x^6 - 2x^5 - x^3 + 7x^2 - 7x + 2$

$-, \quad -, \quad -, \quad -, \quad -, \quad -, \quad +, \quad -, \quad +$

1 2 3

No. of possibilities

Positive roots m	Negative roots k	Total No. of roots n	Imaginary roots
4	3	9	2
2	1	9	6
0	1	9	8

2. Discuss the Maximum possible number of positive and negative zero of the polynomials $x^2 - 5x + 6$ and $x^2 - 5x + 16$. Also draw rough sketch of the graph.

Let $p(x) = x^2 - 5x + 6$

$+, \quad -, \quad +$

1 2

$+$ $\times$

-5 6

-3 -2

$p(x)$ has 2 sign changes

$p(-x) = x^2 + 5x + 6$

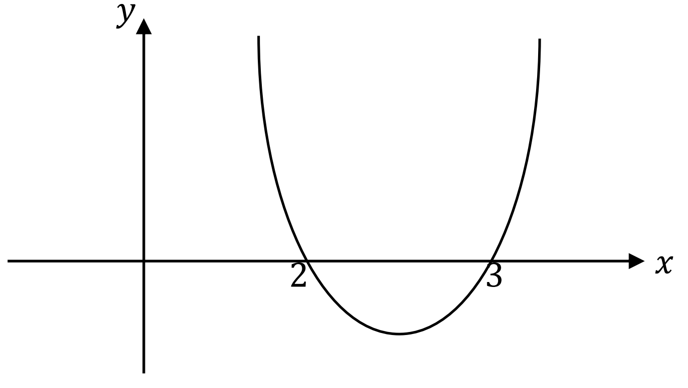
$+, \quad +, \quad +$

$p(-x)$ has no changes

$$x^2-5x+6=0$$
$$(x-3)(x-2)=0$$
$$x-3=0,\ x-2=0$$
$$x=3,\ \ x=2$$

No. of possibilities

Positive roots M	Negative roots k	Total No. of roots n	Imaginary roots
2	0	2	0
0	0	2	2



$$Q(x)=x^2-5x+16$$
$+,\quad -,\quad +$

1 2

$Q(x)$ has 2 sign changes

$$Q(-x)=x^2+5x+16$$
$+,\quad +,\quad +$

$Q(-x)$ has no sign changes

No. of possibilities

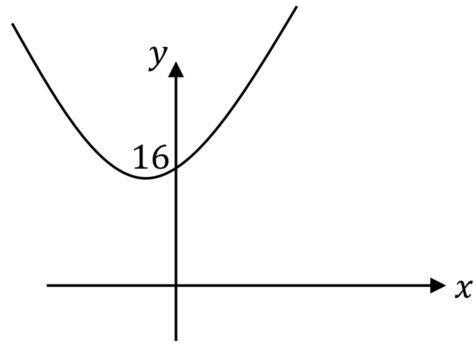
Positive roots M	Negative roots k	Total No. of roots n	Imaginary roots
2	0	2	0
0	0	2	2

$$x^2-5x+16=0$$
$$a=1, b=-5, c=16$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(16)}}{2} = \frac{5 \pm \sqrt{25-64}}{2}$$

$$x = \frac{5 \pm \sqrt{-39}}{2} = \frac{5 \pm i\sqrt{39}}{2}$$

The roots are imaginary hence the does not intersect $x - axis$



3. Show that the equation $x^9 - 5x^5 + 4x^4 + 2x^2 + 1 = 0$ has atleast 6 imaginary solution.

Let $p(x) = x^9 - 5x^5 + 4x^4 + 2x^2 + 1$

$+, \quad -, \quad +, \quad +, \quad +$

1   2

$p(x)$ has 2 sign changes.

$p(-x) = -x^9 + 5x^5 + 4x^4 + 2x^2 + 1$

$-, \quad +, \quad +, \quad +, \quad +$

1

$p(-x)$ has 1 sign changes.

No. of possibilities

Positive roots M	*Negative roots k*	*Total No. of roots n*	*Imaginary roots*
2	1	9	6
0	1	9	8

4. Determine the number of positive and negative roots of the equation $x^9 - 5x^8 - 14x^7 = 0$

Let $p(x) = x^9 - 5x^8 - 14x^7$

$+, \quad -, \quad -$

1

$p(x)$ has 1 sign changes

$p(-x) = -x^9 - 5x^8 + 14x^7$

$-, \quad -, \quad +$

1

The number of possibilities

Positive roots (m)	*Negative roots (k)*	*Total No. of roots n*	*Imaginary roots*
1	1	9	7

5. Find the exact number of real roots and imaginary of the equation $x^9 + 9x^7 + 7x^5 + 5x^3 + 3x$

$Let\ p(x) = x^9 + 9x^7 + 7x^5 + 5x^3 + 3x$

$+, \quad +, \quad +, \quad +, \quad +$

$p(x)$ has no sign changes.

$p(-x) = -x^9 - 9x^7 - 7x^5 - 5x^3 - 3x$

$-, \quad -, \quad -, \quad -, \quad -$

$p(-x)$ has no sign changes.

The number of possibilities

Positive roots (m)	Negative roots (k)	Total No. of roots n	Imaginary roots
0	0	9	9

Inverse Trigonometric Functions

Exercise: 4.1

The inverse sine function and its properties

The **inverse sine function** is defined by

$y = \sin^{-1}x$ or $\arcsin x$ if and only if $\sin y = x$.

The domain of $y = \arcsin x$ is $[-1, 1]$.

The range of $y = \arcsin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$

$\sin^{-1}: [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is called the **principal domain** of sine function and the values $y = \sin^{-1}x$ $[-1, 1]$ are known as **principal values** of the function $y = \sin^{-1}x$

Amplitude and Period of a graph

The **amplitude** of $y = a\sin x$

$$\textbf{amplitude} = |a|$$

The **period** of $y = a\sin bx$ is $\frac{2\pi}{|b|}$

To solve the equation $\sin x = \frac{1}{2}$

To find x : $x = \sin^{-1}\left(\frac{1}{2}\right)$

one has to find all values of x in the interval $(-\infty, \infty)$

one has to find unique value of x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Example: 4.1 Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ ***in radian and degree***

Let $y = \sin^{-1}\left(-\frac{1}{2}\right)$

$\sin y = -\frac{1}{2}$

$\sin y = \sin(-30^\circ)$

$30^\circ = 30 \times \frac{\pi}{180}$ (30 and 180 cancelled: 180 → 6)

$= \frac{\pi}{6}$

$$y = -30^\circ = -\frac{\pi}{6} \quad Since\, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

The principal value of $sin^{-1}\left(-\frac{1}{2}\right)$ *is* $-\frac{\pi}{6}$ *radius this corresponds to* -30°

Example*: 4.2 *Find the principal value of $\boldsymbol{sin^{-1}(2)}$ ***. if it exist***

Domain of $y = sin^{-1} x$ *is* $[-1,1]$ *and* $2 \notin [-1,1]$

$\therefore sin^{-1}(2)$ *does not exist*

Example*: 4.3 *Find the principal value of $\boldsymbol{(i)\, sin^{-1}\left(\frac{1}{\sqrt{2}}\right)}$

$(i)\, sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$$Let\ y = sin^{-1}\left(\frac{1}{\sqrt{2}}\right) Since\, \frac{1}{\sqrt{2}} \in [-1,1]$$

$$sin\, y = \frac{1}{\sqrt{2}}$$

$$sin\, y = sin\, 45^\circ$$

$$y = 45^\circ = \frac{\pi}{4} \quad Since\, \frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$\boldsymbol{(ii)\, sin^{-1}\left(sin\left(-\frac{\pi}{3}\right)\right)}$

$$sin^{-1}\left(sin\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$$

$$Since -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$\boldsymbol{(iii)\, sin^{-1}\left(sin\left(\frac{5\pi}{6}\right)\right)}$

$$\frac{5\pi}{6} = \frac{5 \times 180^\circ}{6} = 150^\circ$$

$$sin\, 150^\circ = sin(180^\circ - 30^\circ)$$

$$= sin\, 30^\circ = sin\frac{\pi}{6}$$

Since $\frac{5\pi}{6} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$sin^{-1}\left(sin\left(\frac{5\pi}{6}\right)\right) = sin^{-1}\left(sin\left(\pi - \frac{\pi}{6}\right)\right)$$

$$= sin^{-1}\left(sin\left(\frac{\pi}{6}\right)\right)$$

$$= \frac{\pi}{6}\ Since\, \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Example*: 4.4 *Find the domain of $\boldsymbol{sin^{-1}(2 - 3x^2)}$

Domain of $sin^{-1} x$ *is* $[-1,1]$

$$-1 \le 2 - 3x^2 \le 1$$

$$-1 - 2 \le 2 - 3x^2 - 2 \le 1 - 2$$

$$-3 \le -3x^2 \le -1$$

$$\frac{-3}{-3} \le \frac{-3x^2}{-3} \le \frac{-1}{-3} \implies 1 \ge x^2 \ge \frac{1}{3}$$

$$\frac{1}{3} \le x^2 \le 1 \implies \frac{1}{\sqrt{3}} \le |x| \le 1$$

$$x \in \left[-1, -\frac{1}{\sqrt{3}}\right] \cup \left[\frac{1}{\sqrt{3}}, 1\right]$$

$$a \le |x| \le b$$
$$x \in [-b, -a] \cup [a, b]$$

1. Find all the values of x such that (i) $-10\pi \le x \le 10\pi$ and $\sin x = 0$

(ii) $-8\pi \le x \le 8\pi$ and $\sin x = -1$

$$\sin x = 0 \implies x = n\pi, n \in I$$

$$n \in [-10, 10] \implies n = 0, \pm 1, \pm 2, \pm 3, \ldots \pm 10$$

$$\sin\theta = \sin\alpha$$
$$\theta = n\pi + (-1)^n \alpha, n \in Z$$
$$Domain\ for\ sin: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

(ii) $\sin x = -1$

$$\sin x = -1 \implies \sin x = \sin(-90°)$$

$$\sin x = \sin\left(-\frac{\pi}{2}\right)$$

$$\sin\theta = \sin\alpha$$

$$Here\ \theta = x, \alpha = -\frac{\pi}{2}$$

$$x = n\pi + (-1)^n \left(-\frac{\pi}{2}\right) \implies x = n\pi + (-1)^n (-1) \left(\frac{\pi}{2}\right)$$

$$x = n\pi + (-1)^{n+1} \left(\frac{\pi}{2}\right), n \in I$$

$$x = n\pi + (-1)^{n+1} \left(\frac{\pi}{2}\right), n \in I$$

$$n = 0$$

$$x = (0)\pi + (-1)^{0+1} \left(\frac{\pi}{2}\right) = 0 + (-1)^1 \left(\frac{\pi}{2}\right)$$

$$x = -\frac{\pi}{2}$$

$$n = 1$$

$$x = 1\pi + (-1)^{1+1} \left(\frac{\pi}{2}\right) = \pi + (-1)^2 \left(\frac{\pi}{2}\right)$$

$$= \pi + \frac{\pi}{2} = \frac{2\pi + \pi}{2}$$

$$x = \frac{3\pi}{2}$$

$$n = 2$$

$$x = 2\pi + (-1)^{2+1} \left(\frac{\pi}{2}\right) = 2\pi + (-1)^3 \left(\frac{\pi}{2}\right)$$

$$= 2\pi - \frac{\pi}{2} = \frac{4\pi - \pi}{2}$$

$x = \frac{3\pi}{2}$

$n = 3 \Rightarrow x = \frac{7\pi}{2}$

$n = 4 \Rightarrow x = \frac{7\pi}{2}$

$n = 5 \Rightarrow x = \frac{11\pi}{2}$

$n = 6 \Rightarrow x = \frac{11\pi}{2}$

$x = (4n-1)\frac{\pi}{2} \ , n = 0, \pm 1, \pm 2, \pm 3, 4$

$\boldsymbol{Ex: 2\ Find\ the\ period\ and\ amplitude\ of}$

$(i)\ y = sin\, 7x$

$y = a\ sinbx$

$Here\ a = 1\ and\ b = 7$

$Amplitude = |a| = |1| = 1$

$period = \frac{2\pi}{|b|} = \frac{2\pi}{|7|} = \frac{2\pi}{7}$

$\boldsymbol{(ii)\ y = -sin\left(\frac{1}{3}x\right)}$

$y = a\ sinbx$

$Here\ a = -1\ and\ b = \frac{1}{3}$

$Amplitude = |a| = |-1| = 1$

$period = \frac{2\pi}{|b|} = \frac{2\pi}{\left|\frac{1}{3}\right|} = \frac{2\pi}{\frac{1}{3}} = 2\pi \times \frac{3}{1}$

$= 6\pi$

$\boldsymbol{(iii)\ y = 4\, sin(-2x)}$

$y = a\ sinbx$

$Here\ a = 4\ and\ b = -2$

$Amplitude = |a| = |4| = 4$

$periods = \frac{2\pi}{|b|} = \frac{2\pi}{|-2|} = \frac{2\pi}{2} = \pi$

$\boldsymbol{3. Sketch\ the\ graph\ of\ y = sin\left(\frac{1}{3}x\right)}$

$Amplitude = |a| = |1| = 1$

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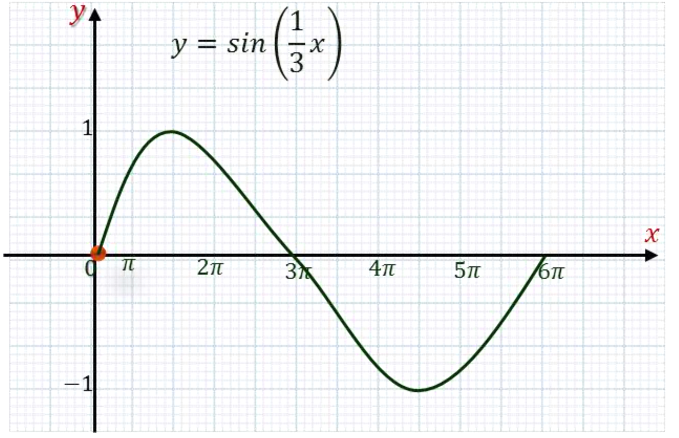
$$period = \frac{2\pi}{|b|} = \frac{2\pi}{\left|\frac{1}{3}\right|} = \frac{2\pi}{\frac{1}{3}} = 2\pi \times \frac{3}{1}$$
$$= 6\pi$$



$$y = sin\left(\frac{1}{3}x\right)$$

$$when\ x = 0, \quad y = sin(0)$$
$$y = 0$$

$$when\ x = 3\pi,\ y = sin\left(\frac{1}{3} \times 3\pi\right)$$
$$y = sin(\pi)$$
$$y = 0$$

Ex: 4 Find the value of (i) $\boldsymbol{sin^{-1}\left[sin\left(\frac{2\pi}{3}\right)\right]}$

$Let\ y = sin^{-1}\left[sin\left(\frac{2\pi}{3}\right)\right]$ $\quad Since\ \frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\frac{2\pi}{3} = \frac{2 \times 180^{\circ}}{3} = 120^{\circ}$$

$$y = sin^{-1}\left[sin\left(\pi - \frac{\pi}{3}\right)\right]$$

$$sin\,120^{\circ} = sin(180^{\circ} - 60^{\circ})$$
$$sin\frac{2\pi}{3} = sin\,60^{\circ} = sin\frac{\pi}{3}$$

$$y = sin^{-1}\left[sin\frac{\pi}{3}\right]$$

$$y = \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

(ii) $\boldsymbol{sin^{-1}\left[sin\left(\frac{5\pi}{4}\right)\right]}$

$Let\ y = sin^{-1}\left[sin\left(\frac{5\pi}{4}\right)\right]$ $\quad Since\ \frac{5\pi}{4} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$y = sin^{-1}\left[sin\left(\pi + \frac{\pi}{4}\right)\right]$$

$$\frac{5\pi}{4} = \frac{5 \times 180^{\circ}}{4} = 5 \times 45^{\circ}$$
$$= 225^{\circ}$$

$$y = sin^{-1}\left[-sin\frac{\pi}{4}\right]$$

$$sin\,225^{\circ} = sin(180^{\circ} + 45^{\circ})$$
$$= -sin\,45^{\circ} = -sin\frac{\pi}{4}$$

$$y = sin^{-1}\left[sin\left(-\frac{\pi}{4}\right)\right]$$

$$sin(-\theta) = -sin\theta$$

$$y = -\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

5. For what value of $\boldsymbol{x}$ **does** $\boldsymbol{sin\,x = sin^{-1}x}$**?**

The only solutions

$$sin\,0 = 0 \quad and\ sin^{-1}0 = 0$$
$$\therefore sin\,0 = sin^{-1}0$$

Hence $x = 0$

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6. Find the domain of the following $f(x) = \sin^{-1}\left(\frac{x^2+1}{2x}\right)$

The Domain of $\sin^{-1} x$ *is* $[-1,1]$

$$-1 \le \frac{x^2+1}{2x} \le 1$$

$$-1 \le \frac{x^2+1}{2x}, \quad \frac{x^2+1}{2x} \le 1$$

$$-2x \le x^2+1, \quad x^2+1 \le 2x$$

$$0 \le x^2+1+2x, \quad x^2+1-2x \le 0$$

$$x^2+2x+1 \ge 0, \quad x^2-2x+1 \le 0$$

$$(x+1)^2 \ge 0, \quad (x-1)^2 \le 0$$

$$x+1 \ge 0, \qquad x-1 \le 0$$

$$x \ge -1, \ldots (1) \qquad x \le 1 \ldots (2)$$

From (1) *and* (2) *Domain* $= [-1,1]$

(ii) $g(x) = 2\sin^{-1}(2x-1) - \frac{\pi}{4}$

$$-1 \le 2x-1 \le 1$$

$$-1+1 \le 2x-1+1 \le 1+1$$

$$0 \le 2x \le 2$$

$$\div 2$$

$$0 \le x \le 1$$

Domain $= [0,1]$

7. Find the value of $\sin^{-1}\left[\sin\frac{5\pi}{9}\cos\frac{\pi}{9} + \cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right]$

Let $y = \sin^{-1}\left[\sin\frac{5\pi}{9}\cos\frac{\pi}{9} + \cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right]$

$\sin A \quad \cos B \quad \cos A \quad \sin B$

$\boxed{\sin(A+B) = \sin A\cos B + \cos A \sin B}$

$$y = \sin^{-1}\left[\sin\left(\frac{5\pi}{9}+\frac{\pi}{9}\right)\right] = \sin^{-1}\left[\sin\left(\frac{6\pi}{9}\right)\right]$$

$$y = \sin^{-1}\left(\sin\frac{2\pi}{3}\right) \quad Since\ \frac{2\pi}{3} \notin \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$\therefore y = \sin^{-1}\left[\sin\left(\pi - \frac{\pi}{3}\right)\right] \quad \therefore y = \sin^{-1}\left[\sin\frac{\pi}{3}\right]$$

$$y = \frac{\pi}{3} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$\frac{2\pi}{3} = \frac{2\times 180^\circ}{3} = 2\times 60^\circ = 120^\circ$$

$$\sin 120^\circ = \sin(180^\circ - 60^\circ) = \sin 60^\circ = \sin\frac{\pi}{3}$$

Exercise: 4.2

Example 4.5: Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

$$Let\ y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \Rightarrow \cos y = \frac{\sqrt{3}}{2} \text{ Since } \frac{\sqrt{3}}{2} \in [-1,1]$$

$$\cos y = \cos 30° \Rightarrow \cos y = \cos\frac{\pi}{6}$$

$$\therefore y = \frac{\pi}{6} \quad since\ \frac{\pi}{6} \in [0,\pi].$$

$$Thus, the\ principal\ value\ of\ \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\ is\ \frac{\pi}{6}$$

Example 4.6: Find $(i)\ \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ $(ii)\ \cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$ $(iii)\ \cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$

$$\cos^{-1} x : [-1,1] \to [0,\pi]$$

$$(i)\ \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$

$$Let\ y = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) \Rightarrow \cos y = -\frac{1}{\sqrt{2}}$$

$$\cos y = \cos\frac{3\pi}{4}$$

$$y = \frac{3\pi}{4} \quad since\ \frac{3\pi}{4} \in [0,\pi]$$

$$\cos(180° - \theta) = -\cos\theta$$

$$\cos 45° = \frac{1}{\sqrt{2}}$$

$$\cos(180 - 45°) = -\frac{1}{\sqrt{2}}$$

$$\cos(135°) = -\frac{1}{\sqrt{2}}$$

$$\frac{\cancel{135}^{\cancel{27}\,3}° \times \pi}{\cancel{180}°_{\cancel{36}\,4}} = \frac{3\pi}{4}$$

$$(ii)\ \cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$$

$$Let\ y = \cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)$$

$$y = \frac{\pi}{3}, \quad since\ \frac{\pi}{3} \in [0,\pi]$$

$$(iii)\ \cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$$

$$Let\ y = \cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right) \quad since\ \frac{7\pi}{6} \notin [0,\pi]$$

$$y = \cos^{-1}\left(\cos\left(2\pi - \frac{5\pi}{6}\right)\right)$$

$$\frac{7\pi}{6} = \frac{7 \times 180^\circ}{6} = 7 \times 30° = 210°$$

$$\cos 210^\circ = \cos(360^\circ - 150^\circ) = \cos 150^\circ = \cos\frac{5\pi}{6}$$

$$y = cos^{-1}\left(cos\left(\frac{5\pi}{6}\right)\right) \quad since \ \frac{5\pi}{6} \in [0,\pi]$$

$$\cancel{150°}^{\cancel{30}\,5} \times \frac{\pi}{\cancel{180°}_{\cancel{36}\,6}} = \frac{5\pi}{6}$$

$$y = \frac{5\pi}{6}$$

$$since\ cos\left(\frac{7\pi}{6}\right) = cos\left(\pi + \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$= cos\left(\frac{5\pi}{6}\right) \ and \ \frac{5\pi}{6} \in \ [0,\pi]$$

Example 4.7: Find the domain of $\boldsymbol{cos^{-1}\left(\frac{2+sin\,x}{3}\right)}$.

By definition,

the domain of $y = cos^{-1}x$ is $-1 \le x \le 1$ or $|x| \le 1$. This leads to

$$-1 \le \frac{2+sin\,x}{3} \le 1 \text{ which is same as } -3 \le 2 + sin\,x \le 3.$$

So, $-5 \le sin\,x \le 1$ reduces to $-1 \le sin\,x \le 1$, which gives

$$-sin^{-1}(1) \le x \le sin^{-1}(1) \ or \ -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

$$Thus, the\ domain\ cos^{-1}\left(\frac{2+sin\,x}{3}\right) \ is\ \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

1. Find all the values of x such that (i) $-6\pi \le x \le 6\pi$ and $cosx = 0$

(i) $-5\pi \le x \le 5\pi$ and $cosx = 1$

$cosx = 0 \implies x = cos^{-1}0$

$$x = (2n+1)\frac{\pi}{2}, n \in I$$

since $-6\pi \le x \le 6\pi$

$n = 0, \pm1, \pm2, \pm3, \pm4, \pm5$

$cos\theta = cos\alpha$

$\theta = 2n\pi \pm \alpha, n \in \mathbb{Z}$

(ii) $cosx = 1$

$cosx = 1 \implies cos\,x = cos0°$

$cos\theta = cos\alpha$

Here $\theta = x, \alpha = 0$

$x = 2n\pi \pm 0$

$x = 2n\pi, n \in I$

since $-5\pi \le x \le 5\pi$

$n = 0, \pm1, \pm2$

2. $\boldsymbol{State\ the\ reason\ for\ cos^{-1}\left[cos\left(-\frac{\pi}{6}\right)\right] \neq -\frac{\pi}{6}}$

$$cos^{-1}\left[cos\left(-\frac{\pi}{6}\right)\right] \neq -\frac{\pi}{6} \qquad Domain\ of\ cos\ x = [0,\pi]$$

$$Since\ -\frac{\pi}{6} \notin [0,\pi]$$

3. $\boldsymbol{Is\ cos^{-1}(-x) = \pi - cos^{-1}(x)\ true?\ Justify\ your\ answer.}$

Let $y = \cos^{-1}(-x)$ $\qquad$ $\boxed{cos(180° - \theta) = -cos\theta}$

$\cos y = -x \Longrightarrow \ x = -\cos y$

$x = \cos(\pi - y) \Longrightarrow \cos^{-1} x = \pi - y$

$y = \pi - \cos^{-1} x$

$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

$\boldsymbol{Ex}$: **4**: $\boldsymbol{Find\ the\ principal\ value\ of\ cos^{-1}\left(\frac{1}{2}\right).}$

$Let\ \ y = cos^{-1}\left(\frac{1}{2}\right)$

$cos\, y = \frac{1}{2}$ $\quad$ Since $\frac{1}{2} \in [-1,1]$

$cos\, y = cos\, 60^{\circ}$

$y = 60^{\circ} = \frac{\pi}{3} \in [0,\pi]$

$Principal\ value\ of\ \ cos^{-1}\left(\frac{1}{2}\right) is\ \frac{\pi}{3}$

5. $\boldsymbol{Find\ the\ value\ of (i)\ 2\, cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}\left(\frac{1}{2}\right)}$

$$2\, cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}\left(\frac{1}{2}\right)$$

$$= 2\, cos^{-1}(cos\, 60^{\circ}) + sin^{-1}(sin\, 30^{\circ})$$

$$= 2\, cos^{-1}\left(cos\frac{\pi}{3}\right) + sin^{-1}\left(sin\frac{\pi}{6}\right)$$

$$= 2 \times \frac{\pi}{3} + \frac{\pi}{6} = \frac{2\pi}{3} + \frac{\pi}{6}$$

$$= \frac{4\pi + \pi}{6} = \frac{5\pi}{6}$$

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$(ii)\ \boldsymbol{cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}(-1)}$

$$cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}(-1)$$

$$= cos^{-1}\left[cos\frac{\pi}{3}\right] + sin^{-1}\left[sin\left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{3} - \frac{\pi}{2} \quad = \frac{2\pi - 3\pi}{6} \quad = -\frac{\pi}{6}$$

$sin\,90^{\circ} = 1$
$sin(-90^{\circ}) = -1$
$sin\left(-\frac{\pi}{2}\right) = -1$

$(iii)\ \boldsymbol{cos^{-1}\left[cos\frac{\pi}{7}cos\frac{\pi}{17} - sin\frac{\pi}{7}sin\frac{\pi}{17}\right]}$

$$Let\ y = cos^{-1}\left[\underbrace{cos\frac{\pi}{7}}_{cos\,A}\underbrace{cos\frac{\pi}{17}}_{cos\,B} - \underbrace{sin\frac{\pi}{7}}_{sin\,A}\underbrace{sin\frac{\pi}{17}}_{sin\,B}\right]$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$= cos^{-1}\left[cos\left(\frac{\pi}{7} + \frac{\pi}{17}\right)\right] = cos^{-1}\left[cos\frac{17\pi + 7\pi}{119}\right]$$

$$y = \frac{24\pi}{119}$$

6. *Find the domain of* $\boldsymbol{(i)\ f(x) = sin^{-1}\left(\frac{|x|-2}{3}\right) + cos^{-1}\left(\frac{1-|x|}{4}\right)}$

$$f(x) = sin^{-1}\left(\frac{|x|-2}{3}\right) + cos^{-1}\left(\frac{1-|x|}{4}\right)$$

$$Take: sin^{-1}\left(\frac{|x|-2}{3}\right)$$

Domain of $sin^{-1}x$ *is* $[-1,1]$

$$-1 \le \frac{|x|-2}{3} \le 1 \Rightarrow -3 \le |x| - 2 \le 3$$

$$-3 + 2 \le |x| - 2 + 2 \le 3 + 2$$

$$-1 \le |x| \le 5 \Rightarrow \quad |x| \ge -1,\ \ |x| \le 5$$

$$-1 \le x \le 1, -5 \le x \le 5$$

$|x| \ge -1$
$x \ge -1\ or\ -x \ge -1$
$x \ge -1\ or\ x \le 1$
$-1 \le x \le 1$

$|x| \le 5$
$x \le 5\ or\ -x \le 5$
$x \le 5\ or\ x \ge -5$
$-5 \le x \le 5$

$$Take: \cos^{-1}\left(\frac{1-|x|}{4}\right)$$

Domain of $cos^{-1}x$ *is* $[-1,1]$

$$-1 \le \frac{1-|x|}{4} \le 1$$

$$-4 \le 1 - |x| \le 4$$

$$-4-1 \leq 1-|x|-1 \leq 4-1$$

$$-5 \leq -|x| \leq 3 \implies 5 \geq |x| \geq -3$$

$$-3 \leq |x| \leq 5$$

$$|x| \geq -3,\ |x| \leq 5$$

$$x \geq -3 \ or\ -x \geq -3,\ x \leq 5 \ or\ -x \leq 5$$

$$x \geq -3 \ or\ x \leq 3 \quad x \leq 5 \ or\ x \geq -5$$

$$-3 \leq x \leq 3 \qquad -5 \leq x \leq 5$$

Hence the Domain is $[-5,5]$

6. Find the domain of (ii) $\boldsymbol{g(x) = sin^{-1} x + cos^{-1} x}$

$(ii)\ g(x) = sin^{-1} x + cos^{-1} x$

Domain of $sin^{-1} x \quad [-1,1]$

Domain of $cos^{-1} x \quad [-1,1]$

Hence the Domain is $[-1,1]$

7. For what value of x, the inequality $\boldsymbol{\frac{\pi}{2} < cos^{-1}(3x-1) < \pi}$ **holds?**

$$\frac{\pi}{2} < \cos^{-1}(3x-1) < \pi$$

$$\cos\frac{\pi}{2} < 3x-1 < \cos\pi$$

$$0 < 3x-1 < -1$$

$$0+1 < 3x-1+1 < -1+1$$

$$1 < 3x < 0$$

$$\div 3$$

$$\frac{1}{3} < x < 0$$

8. Find the value of (i) $\boldsymbol{cos\left[cos^{-1}\left(\frac{4}{5}\right) + sin^{-1}\left(\frac{4}{5}\right)\right]}$

$$cos\left[cos^{-1}\left(\frac{4}{5}\right) + sin^{-1}\left(\frac{4}{5}\right)\right]$$

Let $y = \cos^{-1}\left(\frac{4}{5}\right) \implies \cos y = \frac{4}{5}$

$\cos\theta = \sin(90^\circ - \theta)$

$$= cos[y + sin^{-1}(cos\, y)]$$

$$= cos\left[y + sin^{-1}\left(sin\left(\frac{\pi}{2} - y\right)\right)\right]$$

$$= cos\left(y + \frac{\pi}{2} - y\right) = cos\frac{\pi}{2} = 0$$

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$(ii)\ \boldsymbol{cos^{-1}\left[cos\left(\frac{4\pi}{3}\right)\right]+cos^{-1}\left[cos\left(\frac{5\pi}{4}\right)\right]}$

$\frac{4\pi}{3}=\frac{4\times 180^\circ}{3}=4\times 60^\circ=240^\circ$

$\frac{5\pi}{4}=\frac{5\times 180^\circ}{4}=5\times 45^\circ=225^\circ$

$$cos^{-1}\left[cos\left(\frac{4\pi}{3}\right)\right]+cos^{-1}\left[cos\left(\frac{5\pi}{4}\right)\right]$$

$\frac{4\pi}{3}\notin[0,\pi]$ *and* $\frac{5\pi}{4}\notin[0,\pi]$

$cos\,240^\circ=cos(360^\circ-120^\circ)$

$=cos\,120^\circ=cos\frac{2\pi}{3}$

$cos\,225^\circ=cos(360^\circ-135^\circ)$

$=cos\,135^\circ=cos\frac{3\pi}{4}$

$$=cos^{-1}\left(cos\frac{2\pi}{3}\right)+cos^{-1}\left(cos\frac{3\pi}{4}\right)$$

$$=\frac{2\pi}{3}+\frac{3\pi}{4}=\frac{8\pi+9\pi}{12}=\frac{17\pi}{12}$$

$135^\circ\times\frac{\pi}{180^\circ}=\frac{3\pi}{4}$ (cancelling: 135 → 27 → 3; 180 → 36 → 4)

EXERCISE 4.3

Example 4.8: Find the principal value of $\tan^{-1}(\sqrt{3})$

$$Let\ y = \tan^{-1}(\sqrt{3})$$
$$\tan y = \sqrt{3}$$
$$\tan y = \tan 60°$$
$$\tan y = \tan\frac{\pi}{3}$$
$$y = \frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Example 4.9: Find the principal value of (i) $\tan^{-1}(-\sqrt{3})$

$$Let\ y = \tan^{-1}(-\sqrt{3})$$
$$\tan y = -\sqrt{3}$$
$$\tan y = \tan(-60°)$$
$$y = -60°$$
$$y = -\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Example 4.8: Find the principal value of $\tan^{-1}(\tan 60°)$

$$Let\ y = \tan^{-1}(\tan 60°)$$
$$y = 60°$$
$$y = \frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

(ii) $\tan^{-1}\left(\tan\frac{3\pi}{5}\right)$

$$let\ y = \tan^{-1}\left(\tan\frac{3\pi}{5}\right) \quad since\ \frac{3\pi}{5} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$y = \tan^{-1}\left[\tan\left(\pi - \frac{2\pi}{5}\right)\right]$$
$$y = \tan^{-1}\left(-\tan\frac{2\pi}{5}\right)$$
$$y = \tan^{-1}\left[\tan\left(-\frac{2\pi}{5}\right)\right]$$
$$y = -\frac{2\pi}{5} \quad since\ \frac{3\pi}{5} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\frac{3\pi}{5} = \frac{3 \times 180°}{5}\ 36°$$
$$= 3 \times 36 = 108°$$
$$\tan 108° = \tan(180° - 72°)$$
$$= -\tan 72° = -\tan\frac{2\pi}{5}$$

$$72 \times \frac{\pi}{180°} = \frac{2\pi}{5}$$

$(iii)\ \boldsymbol{tan^{-1}(tan\,2019)}$

$let\ y = tan^{-1}(tan\,2019)\ \ since\ 2019 \in \mathbb{R}$

$y = 2019$

$tan\,45° = 1$

$tan(-45°) = -1$

$tan\left(-\frac{\pi}{4}\right) = -1$

$sin(-30°) = -\frac{1}{2}$

$sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

Example 4.10: $\boldsymbol{tan^{-1}(-1) + cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}\left(\frac{-1}{2}\right)}$

$$tan^{-1}(-1) + cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}\left(\frac{-1}{2}\right)$$

$$= tan^{-1}\,tan\left(-\frac{\pi}{4}\right) + cos^{-1}\left(cos\frac{\pi}{3}\right) + sin^{-1}\,sin\left(-\frac{\pi}{6}\right)$$

$$= -\frac{\pi}{4} + \frac{\pi}{3} - \frac{\pi}{6} = \frac{-3\pi + 4\pi - 2\pi}{12} = -\frac{\pi}{12}$$

Example 4.11: **Prove that** $\boldsymbol{tan(sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}, -1 < x < 1}$

$Assume: 0 < x < 1$

$Let\ \theta = sin^{-1}x\ \ then\ \ tan\,\theta = \frac{x}{\sqrt{1-x^2}}$

$\theta = sin^{-1}x$

$$\left.\begin{array}{l} when\ x = 0,\ \theta = 0 \\ when\ x = 1,\ \theta = \frac{\pi}{2} \end{array}\right\} \Longrightarrow Domain:\ 0 < \theta < \frac{\pi}{2}$$

$$tan\,\theta = \frac{x}{\sqrt{1-x^2}}$$

$$\left.\begin{array}{l} when\ \theta = 0,\ x = 0 \\ when\ \theta = \frac{\pi}{2},\ x = \infty \end{array}\right\} \Longrightarrow Range:\ 0 < x < \infty$$

$$x = sin\theta \implies tan\,\theta = \frac{x}{\sqrt{1-x^2}}$$

$Assume: -1 < x < 0$

$Let\ \theta = sin^{-1}x\ \ then\ \ tan\,\theta = \frac{x}{\sqrt{1-x^2}}$

$\theta = sin^{-1}x$

$$\left.\begin{array}{l} when\ x = 0,\ \theta = 0 \\ when\ x = -1,\ \theta = -\frac{\pi}{2} \end{array}\right\} \Longrightarrow Domain: -\frac{\pi}{2} < \theta < 0$$

$$tan\,\theta = \frac{x}{\sqrt{1-x^2}}$$

$$\left.\begin{array}{l} when\ \theta = 0,\ x = 0 \\ when\ \theta = -\frac{\pi}{2},\ x = -\infty \end{array}\right\} \Longrightarrow Range: -\infty < x < 0$$

$$x = sin\theta \implies tan\,\theta = \frac{x}{\sqrt{1-x^2}}$$

$$-1 < x < 1$$

$$tan(sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$$

$$Domain\ of\ tan: -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$Range\ of\ tan: (-\infty, \infty)$$

$$tan(sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}, -1 < x < 1$$

1(i). Find the domain of the following functions $tan^{-1}\left(\sqrt{9-x^2}\right)$

$(i)\ tan^{-1}\left(\sqrt{9-x^2}\right)$

$Domain\ of\ tan^{-1}x : \mathbb{R}\ or\ (-\infty, \infty)$

Hence x must be real number

To find the values of x

The expression $\sqrt{9-x^2}$ *is real*

$$\sqrt{9-x^2} \geq 0 \implies 9 - x^2 \geq 0$$

$$3^2 - x^2 \geq 0$$

$$(3+x)(3-x) \geq 0 \implies 3 + x \geq 0, 3 - x \geq 0$$

$$x \geq -3, -x \geq -3$$

$$x \leq 3$$

$$-3 \leq x \leq 3$$

Hence the domain of the function is $[-3, 3]$

1(ii). Find the domain of the following functions $\frac{1}{2}tan^{-1}(1-x^2) - \frac{\pi}{4}$

$$\frac{1}{2}tan^{-1}(1-x^2) - \frac{\pi}{4}$$

$Domain\ of\ tan^{-1}x : \mathbb{R}\ or\ (-\infty, \infty)$

Hence x must be real number. Here $1 - x^2$ *is real for all values of x*

Hence the domain of the function is in $\mathbb{R}$

2(i). Find the value of $tan^{-1}\left(tan\frac{5\pi}{4}\right)$

$Let\ y = tan^{-1}\left(tan\frac{5\pi}{4}\right)\quad since\ \frac{5\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$tan(180° + \theta) = tan\theta$

$$= tan^{-1}\left[tan\left(\pi+\frac{\pi}{4}\right)\right] \ = tan^{-1}\left(tan\frac{\pi}{4}\right)$$

$$=\frac{\pi}{4}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

2(ii). Find the value of $\boldsymbol{tan^{-1}\left[tan\left(-\frac{\pi}{6}\right)\right]}$

$$Let\ y = tan^{-1}\left[tan\left(-\frac{\pi}{6}\right)\right]$$

$$=-\frac{\pi}{6}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

3. Find the value of (i) $\boldsymbol{tan\left[tan^{-1}\left(\frac{7\pi}{4}\right)\right]}$

$$Let\ y = tan\left[tan^{-1}\left(\frac{7\pi}{4}\right)\right]$$

$$y=\frac{7\pi}{4}\in\mathbb{R}$$

$$(ii)\ tan[tan^{-1} 1947]$$

$$Let\ y = tan[tan^{-1} 1947]$$

$$y = 1947$$

$$(ii)\ tan[tan^{-1}(-0.2021)]$$

$$Let\ y = tan[tan^{-1}(-0.2021)]$$

$$y = -0.2021$$

4. Find the value of (i) $\boldsymbol{tan\left[cos^{-1}\left(\frac{1}{2}\right)-sin^{-1}\left(-\frac{1}{2}\right)\right]}$

$$= tan\left[cos^{-1}\left(\frac{1}{2}\right)-sin^{-1}\left(-\frac{1}{2}\right)\right]$$

$$sin\left(-\frac{\pi}{6}\right)=-\frac{1}{2}$$

$$= tan\left[cos^{-1}\left(cos\frac{\pi}{3}\right)-sin^{-1}\left(sin\left(-\frac{\pi}{6}\right)\right)\right]$$

$$= tan\left(\frac{\pi}{3}+\frac{\pi}{6}\right) \ = tan\frac{\pi}{2}=\infty$$

4. Find the value of (ii) $\boldsymbol{sin\left[tan^{-1}\left(\frac{1}{2}\right)-cos^{-1}\left(\frac{4}{5}\right)\right]}$

$$sin\left[tan^{-1}\left(\frac{1}{2}\right)-cos^{-1}\left(\frac{4}{5}\right)\right]$$

$$Let\ x = tan^{-1}\frac{1}{2}, y = cos^{-1}\left(\frac{4}{5}\right)$$

$$\tan x = \frac{1}{2} \quad , \cos y = \frac{4}{5}$$

$$\boxed{\sin(x-y) = \sin x \cos y - \cos x \sin y}$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \left(\frac{1}{2}\right)^2 = \sec^2 x$$

$$1 + \frac{1}{4} = \sec^2 x$$

$$\frac{5}{4} = \sec^2 x \implies \sec^2 x = \frac{5}{4}$$

$$\sec x = \sqrt{\frac{5}{4}} \implies \sec x = \frac{\sqrt{5}}{2}$$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$\cos x = \frac{2}{\sqrt{5}} \implies \sin x = \sqrt{1-\cos^2 x}$$

$$\sin x = \sqrt{1-\left(\frac{2}{\sqrt{5}}\right)^2} \implies \sin x = \sqrt{1-\frac{4}{5}}$$

$$\sin x = \sqrt{\frac{1}{5}} \implies \sin x = \frac{1}{\sqrt{5}}$$

$$\boxed{\sin y = \sqrt{1-\cos^2 y}}$$

$$\sin y = \sqrt{1-\left(\frac{4}{5}\right)^2} \implies \sin y = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{25-16}{25}}$$

$$\sin y = \sqrt{\frac{9}{25}} \implies \sin y = \frac{3}{5}$$

$$\boxed{\sin(x-y) = \sin x \cos y - \cos x \sin y}$$

$$\sin(x-y) = \frac{1}{\sqrt{5}} \times \frac{4}{5} - \frac{2}{\sqrt{5}} \times \frac{3}{5} = \frac{4}{5\sqrt{5}} - \frac{6}{5\sqrt{5}}$$

$$= \frac{4-6}{5\sqrt{5}} = \frac{-2}{5\sqrt{5}} = -\frac{2 \times \sqrt{5}}{5\sqrt{5} \times \sqrt{5}} = \frac{-2\sqrt{5}}{5 \times 5}$$

$$\sin(x-y) = \frac{-2\sqrt{5}}{25}$$

$$\text{where } x = \tan^{-1}\frac{1}{2}, y = \cos^{-1}\left(\frac{4}{5}\right)$$

$$\sin\left[\tan^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right] = \frac{-2\sqrt{5}}{25}$$

(iii) $\boldsymbol{\cos\left(\sin^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right)}$

$$\text{Let } x = \sin^{-1}\frac{4}{5}, y = \tan^{-1}\frac{3}{4}$$

$$\sin x = \frac{4}{5}, \tan y = \frac{3}{4}$$

$$\boxed{\cos(x-y) = \cos x \cos y + \sin x \sin y}$$

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$$\cos x = \sqrt{1-\sin^2 x}$$

$$\cos x = \sqrt{1-\left(\frac{4}{5}\right)^2} \implies \cos x = \sqrt{1-\frac{16}{25}}$$

$$\cos x = \sqrt{\frac{9}{25}} \implies \cos x = \frac{3}{5}$$

$$\cos y = \frac{4}{5}$$

$$\sin y = \sqrt{1-\left(\frac{4}{5}\right)^2}$$

$$\sin y = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{25-16}{25}}$$

$$\sin y = \sqrt{\frac{9}{25}} \implies \sin y = \frac{3}{5}$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$= \frac{3}{5}\times\frac{4}{5}+\frac{4}{5}\times\frac{3}{5} = \frac{12}{25}+\frac{12}{25} = \frac{24}{25}$$

$$\cos\left(\sin^{-1}\left(\frac{4}{5}\right)-\tan^{-1}\left(\frac{3}{4}\right)\right) = \frac{24}{25}$$

$$1+\tan^2 y = \sec^2 y$$

$$1+\left(\frac{3}{4}\right)^2 = \sec^2 y$$

$$1+\frac{9}{16} = \sec^2 y$$

$$\frac{16+9}{16} = \sec^2 y \implies \sec^2 y = \frac{25}{16}$$

$$\sec y = \sqrt{\frac{25}{16}} \implies \sec y = \frac{5}{4}$$

$$\cos y = \frac{4}{5}$$

$$\sin y = \sqrt{1-\cos^2 y}$$

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The Cosecant Function and the Inverse Cosecant Function

$Domain\ of\ cosecant\ functions\ is\ the\ set\ of$
$all\ real\ numbers, except\ integer\ multiples\ of\ \pi$

$$\mathbb{R}-\{n\pi, n\in\mathbb{Z}\} \Rightarrow \left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$$

$Range\ of\ cosecant\ is\ (-\infty,-1] \cup [1,\infty)$

$y = cosecx\ does\ not\ take\ any\ value\ in$
$between\ -1\ and\ 1.$

$Range\ of\ cosecant: \mathbb{R}-(-1,1)$

Periodic Properties:

$$period = 2\pi$$

$$cosec(x+2\pi) = cosecx$$

odd functions: $cosec(-x) = -cosecx$

Inverse Cosecant Function

$Domain: \mathbb{R}-(-1,1)$

$Range: \left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

$y = cosecx\ does\ not\ take\ any\ value\ in$
$between\ -1\ and\ 1.$

$Range\ of\ cosecant: \mathbb{R}-(-1,1)$

The secant Function and the Inverse secant Function

- $Domain\ of\ secant\ functions\ is\ the\ set\ of$
$all\ real\ numbers, except\ odd\ integer\ multiples\ of\ \frac{\pi}{2}$

$$\mathbb{R}-\left\{(2n+1)\frac{\pi}{2}, n\in\mathbb{Z}\right\} \Rightarrow [0,\pi]-\left\{\frac{\pi}{2}\right\}$$

$Range\ of\ secant: \mathbb{R}-(-1,1)$

$$secant\ have\ period\ 2\pi$$

$$sec(x+2\pi) = secx$$

secant are even functions

$$sec(-x) = secx$$

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EXERCISE 4.4

Inverse secant Function

$Domain : \mathbb{R} - (-1,1)$

$Range : [0,\pi] - \left\{\frac{\pi}{2}\right\}$

The cotangent function

$Domain : [0,\pi]$

$Range : \mathbb{R}$

$cotangent\ have\ period\ \pi$

$cot(x+\pi) = cotx$

odd functions : $cot(-x) = -cotx$

Inverse cotangent Function

$Domain : \mathbb{R}$

$Range : [0,\pi]$

***Example*: 4.12** $Find\ the\ principal\ value\ of\ (i)\ cosec^{-1}(-1)$

$Let\ y = cosec^{-1}(-1)$

$cosec\, y = -1 \Longrightarrow \frac{1}{\text{cosec}\, y} = -1$

$sin\, y = -1$

$siny = sin\left(-\frac{\pi}{2}\right)$

$y = -\frac{\pi}{2} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] - \{0\}$

$(ii)\ sec^{-1}(-2)$

$Domain$
$: \mathbb{R} - (-1,1)$

$Let\ y = sec^{-1}(-2)$

$sec\, y = -2$

$\frac{1}{sec\, y} = \frac{1}{-2} \Longrightarrow \quad cos\, y = -\frac{1}{2}$

$cos\, y = cos\frac{2\pi}{3}$

$y = \frac{2\pi}{3} \in [0,\pi] - \left\{\frac{\pi}{2}\right\}$

$cos\, x = \frac{1}{sec\, x}$

$cos\, 60° = \frac{1}{2}$

$cos(180 - 60°) = -\frac{1}{2}$

$cos\, 120° = -\frac{1}{2}$

$120^{\circ} \times \frac{\pi}{180^{\circ}} = \frac{2\pi}{3}$

Example: 4.13: Find the value of $sec^{-1}\left(\frac{-2\sqrt{3}}{3}\right)$

$$Let\ y = sec^{-1}\left(\frac{-2\sqrt{3}}{3}\right) = sec^{-1}\left(\frac{-2\sqrt{3}}{\sqrt{3}\times\sqrt{3}}\right)$$

$$y = sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) \Longrightarrow \quad sec\,y = -\frac{2}{\sqrt{3}}$$

$$\frac{1}{sec\,y} = -\frac{\sqrt{3}}{2} \Longrightarrow cos\,y = -\frac{\sqrt{3}}{2}$$

$$cos\,y = cos\left(\frac{5\pi}{6}\right)$$

$$y = \frac{5\pi}{6} \in [0,\pi] - \left\{\frac{\pi}{2}\right\}$$

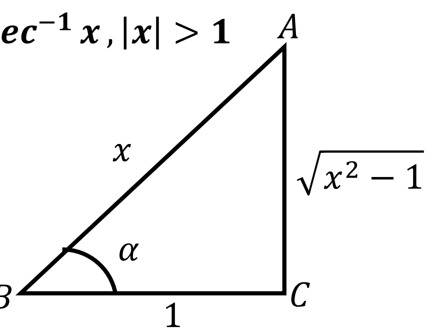
$$cos\,30° = \frac{\sqrt{3}}{2}$$

$$cos(180° - 30°) = -\frac{\sqrt{3}}{2}$$

$$cos\,150° = -\frac{\sqrt{3}}{2}$$

$$\frac{150°}{30}{}^{5} \times \frac{\pi}{180°}{}_{36\ 6} = \frac{5\pi}{6}$$

Example: 4.13 Show that $cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = sec^{-1}x, |x| > 1$



$$Let\ \alpha = cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$$

$$cot\alpha = \frac{1}{\sqrt{x^2-1}} \Longrightarrow tan\alpha = \frac{\sqrt{x^2-1}}{1}$$

$$sec\alpha = \frac{hyp}{adj} \Longrightarrow sec\alpha = \frac{x}{1}$$

$$sec\alpha = x \quad \Longrightarrow \alpha = \sec^{-1}x$$

$$cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1}x$$

$$AB^2 = BC^2 + AC^2$$

$$AB^2 = 1^2 + \left(\sqrt{x^2-1}\right)^2$$

$$AB^2 = 1 + x^2 - 1$$

$$AB^2 = x^2 \Longrightarrow AB = \sqrt{x^2}$$

$$AB = x$$

1. Find the principal value of (i) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

$$Let\ y = sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

$$sec\,y = \frac{2}{\sqrt{3}} \Longrightarrow \frac{1}{sec\,y} = \frac{\sqrt{3}}{2}$$

$$cos\,y = \frac{\sqrt{3}}{2} \Longrightarrow \quad cos\,y = cos\,30°$$

$$cos\,y = cos\frac{\pi}{6}$$

$$y = \frac{\pi}{6} \in [0,\pi] - \left\{\frac{\pi}{2}\right\}$$

$ii)$ $\mathbf{cosec^{-1}(-\sqrt{2})}$

Let $y = \text{cosec}^{-1}(-\sqrt{2})$

$$\text{cosec}\, y = -\sqrt{2} \Rightarrow \frac{1}{cosec\, y} = -\frac{1}{\sqrt{2}}$$

$$sin\, y = \frac{-1}{\sqrt{2}} \Rightarrow sin\, y = sin(-45°)$$

$$sin\, y = sin\left(-\frac{\pi}{4}\right)$$

$$y = -\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

$$sin\, 45° = \frac{1}{\sqrt{2}}$$

$$sin(-45°) = \frac{1}{\sqrt{2}}$$

(iii) $\mathbf{cot^{-1}\sqrt{3}}$

$$y = cot^{-1}(\sqrt{3})$$

$$cot\, y = \sqrt{3} \Rightarrow \frac{1}{\cot y} = \frac{1}{\sqrt{3}} \Rightarrow tan\, y = \frac{1}{\sqrt{3}}$$

$$tan\, y = tan 30°$$

$$tan\, y = tan\frac{\pi}{6}$$

$$y = \frac{\pi}{6} \in (0, \pi)$$

2. *Find the value of* (i) $\mathbf{tan^{-1}\sqrt{3} - sec^{-1}(-2)}$

(i) $\mathbf{tan^{-1}\sqrt{3} - sec^{-1}(-2)}$

$$Let\ x = tan^{-1}\sqrt{3} \Rightarrow tan\, x = \sqrt{3}$$

$$tan\, x = tan 60° \Rightarrow tan\, x = tan\frac{\pi}{3}$$

$$x = \frac{\pi}{3}$$

$$Let\ y = sec^{-1}(-2)$$

$$sec\, y = -2 \Rightarrow \frac{1}{sec\, y} = \frac{-1}{2}$$

$$cos\, y = -\frac{1}{2} \Rightarrow cos\, y = cos\frac{2\pi}{3}$$

$$y = \frac{2\pi}{3}$$

$$\cos 60° = \frac{1}{2}$$

$$\cos(180° - 60) = -\frac{1}{2}$$

$$\cos 120° = -\frac{1}{2}$$

$$120° \times \frac{\pi}{180°} = \frac{2\pi}{3}$$

$$tan^{-1}\sqrt{3} - sec^{-1}(-2) = x - y$$

$$= \frac{\pi}{3} - \frac{2\pi}{3} = \frac{\pi - 2\pi}{3} = -\frac{\pi}{3}$$

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$(ii)\ \boldsymbol{sin^{-1}(-1) + cos^{-1}\left(\frac{1}{2}\right) + cot^{-1}(2)}$

$$Let\ x = sin^{-1}(-1) \Longrightarrow\ sin\,x = -1$$

$$sin\,x = sin\left(-\frac{\pi}{2}\right) \Longrightarrow\ x = -\frac{\pi}{2}$$

$$-90° + 60° = -30°$$
$$= -\frac{\pi}{6}$$

$$Let\ y = cos^{-1}\left(\frac{1}{2}\right) \Longrightarrow\ \cos y = \frac{1}{2}$$

$$\cos y = cos\frac{\pi}{3} \Longrightarrow\ y = \frac{\pi}{3}$$

$$sin^{-1}(-1) + cos^{-1}\left(\frac{1}{2}\right) + cot^{-1}(2) = x + y + \cot^{-1}(2)$$
$$= -\frac{\pi}{2} + \frac{\pi}{3} + \cot^{-1}(2)$$
$$= \cot^{-1}(2) - \frac{\pi}{6}$$

$iii)\ \boldsymbol{cot^{-1}(1) + sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) - sec^{-1}(-\sqrt{2})}$

$$Let\ x = cot^{-1}(1) \Longrightarrow\ cot\,x = 1$$

$$\frac{1}{cot\,x} = 1 \Longrightarrow\ tanx = 1$$

$$tanx = tan\frac{\pi}{4} \Longrightarrow\ x = \frac{\pi}{4}$$

$$y = sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) \Longrightarrow\ sin\,y = -\frac{\sqrt{3}}{2}$$

$$sin\,y = sin\left(-\frac{\pi}{3}\right) \Longrightarrow\ y = -\frac{\pi}{3}$$

$$z = sec^{-1}(-\sqrt{2}) \Longrightarrow\ sec\,z = -\sqrt{2}$$

$$\frac{1}{sec\,z} = -\frac{1}{\sqrt{2}} \Longrightarrow\ cos\,z = -\frac{1}{\sqrt{2}}$$

$$cos\,z = -\frac{1}{\sqrt{2}} \Longrightarrow\ cos\,z = cos\left(\frac{3\pi}{4}\right)$$

$$z = \frac{3\pi}{4}$$

$$cot^{-1}(1) + sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) - sec^{-1}(-\sqrt{2}) = x + y - z$$

$$= \frac{\pi}{4} + \left(-\frac{\pi}{3}\right) - \frac{3\pi}{4}$$

$$= \frac{\pi}{4} - \frac{\pi}{3} - \frac{3\pi}{4}$$

$$= \frac{3\pi - 4\pi - 9\pi}{12}$$

$$= \frac{-\pi - 9\pi}{12} = -\frac{10\pi}{12} = -\frac{5\pi}{6}$$

$$\cos 45° = \frac{1}{\sqrt{2}}$$

$$cos(180° - 45°) = -\frac{1}{\sqrt{2}}$$

$$\cos 135° = -\frac{1}{\sqrt{2}}$$

$$\overset{3}{\cancel{138}}° \times \frac{\pi}{\underset{4}{\cancel{180}}°} = \frac{3\pi}{4}$$

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Circle

Definition:

A circle is the locus of a point which moves in such a way that its distance from a fixed point is always constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

The equation of $\boldsymbol{circle\ (x-h)^2 + (y-k)^2 = r^2}$

$\boldsymbol{x^2 + y^2 = r^2}$ ***passing through the origin.***

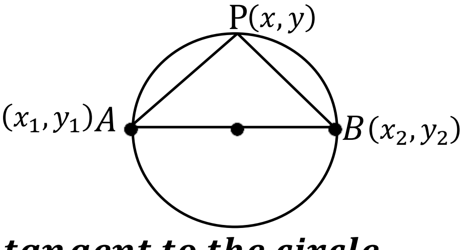
$$General\ equation\ of\ circle : x^2 + y^2 + 2gx + 2fy + c = 0$$

$$Centre : (-g, -f)$$

$$Radius : r = \sqrt{g^2 + f^2 - c}$$

$The\ general\ second\ degree\ equation$
$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0\ represents\ a\ circle$
$if\ (i)\ a = b\ i.e.\ coefficient\ of\ x^2 = coefficient\ of\ y^2$
$(ii)\ h = 0\ i.e.\ no\ xy\ term$

$The\ equation\ of\ a\ circle\ with\ (x_1, y_1)\ and\ (x_2, y_2)\ as\ extremities$
$of\ the\ diameters\ is\ (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$



$Condition\ for\ the\ line\ y = mx + c\ to\ be\ a\ tangent\ to\ the\ circle$
$x^2 + y^2 = a^2\ and\ finding\ the\ point\ of\ contact$

$$c = \pm a\sqrt{m^2 + 1}$$

$$Point\ of\ contact \left(\frac{-am}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right) or \left(\frac{am}{\sqrt{m^2+1}}, \frac{-a}{\sqrt{m^2+1}}\right)$$

$$Equation\ of\ tangent\ to\ the\ circle\ y = mx \pm a\sqrt{m^2 + 1}$$

Tangent and Normal

$Equation\ of\ tangent\ at\ (x_1, y_1)\ to\ the\ circle\ x^2 + y^2 + 2gx + 2fy + c = 0$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

$Equation\ of\ normal\ at\ (x_1, y_1)\ to\ the\ circle\ x^2 + y^2 + 2gx + 2fy + c = 0$

$$yx_1 - xy_1 + g(y - y_1) - f(x - x_1) = 0$$

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Note :

i) *Equation of tangent at* (x_1, y_1) *to the circle* $x^2 + y^2 = r^2$ *with centre* $(0,0)$

$$\boldsymbol{xx_1 + yy_1 = r^2}$$

ii) *Equation of normal at* (x_1, y_1) *to the circle with centre* $(0,0)$

$$\boldsymbol{yx_1 - xy_1 = 0}$$

Example 5.1: Find the general equation of a circle with centre $(-3,-4)$ and radius is 3 units.

Centre $(h,k) = (-3,-4)$ *and radius* : $r = 3$

Equation of the circle : $(x-h)^2 + (y-k)^2 = r^2$

$(x+3)^2 + (y+4)^2 = 3^2 \implies x^2 + 2(x)(3) + 3^2 + y^2 + 2(y)(4) + 4^2 = 9$

$x^2 + 6x + 9 + y^2 + 8y + 16 = 9 \implies x^2 + y^2 + 6x + 8y + 25 - 9 = 0$

$$x^2 + y^2 + 6x + 8y + 16 = 0$$

(OR)

General equation: $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre is $(-g, -f) = (3,4)$ *and radius* : $r = 4$

Radius: $r = \sqrt{g^2 + f^2 - c}$

$3^2 = 3^2 + 4^2 - c \implies 9 = 9 + 16 - c \implies 0 = 16 - c \implies \boxed{c = 16}$

$x^2 + y^2 + 2(3)x + 2(4)y + 16 = 0$

$x^2 + y^2 + 6x + 8y + 16 = 0$

Example 5.2: Find the equation of the circle described on the chord $3x + y + 5 = 0$ of circle $x^2 + y^2 = 16$ as diameter

Circle: $x^2 + y^2 - 16 = 0$ *and Line*: $3x + y + 5 = 0$

$x^2 + y^2 - 16 + \lambda(3x + y + 5) = 0$

$x^2 + y^2 + 3\lambda x + y\lambda + 5\lambda - 16 = 0$

Equation of circle : $x^2 + y^2 + \boxed{3\lambda}x + \boxed{\lambda}y + \boxed{5\lambda - 16} = 0$

Compare with $x^2 + y^2 + \boxed{2g}x + \boxed{2f}y + \boxed{c} = 0$

$2g = 3\lambda, 2f = \lambda, c = 5\lambda - 16$

$g = \frac{3\lambda}{2},\ f = \frac{\lambda}{2}$

Centre $= (-g, -f) \implies$ *Centre* $= \left(-\frac{3\lambda}{2}, -\frac{\lambda}{2}\right)$

$\left(\frac{-3\lambda}{2}, \frac{-\lambda}{2}\right)$ *lies on the line* $3x + y + 5 = 0 \implies 3\left(\frac{-3\lambda}{2}\right) - \frac{\lambda}{2} + 5 = 0$

$$\frac{-9\lambda}{2} - \frac{\lambda}{2} + 5 = 0 \Rightarrow \frac{-10\lambda}{2} + 5 = 0 \Rightarrow \frac{-10\lambda}{2} = -5$$

$$-10\lambda = -10 \Rightarrow \boxed{\lambda = 1}$$

Required equation of circle is $x^2 + y^2 - 16 + 1(3x + y + 5) = 0$

$$x^2 + y^2 - 16 + 3x + y + 5 = 0 \Rightarrow x^2 + y^2 + 3x + y - 11 = 0$$

Example 5.3: Determine whether $x + y - 1 = 0$ is the equation of a diameter of the circle $x^2 + y^2 - 6x + 4y + c = 0$ for all possible values of c.

$$x^2 + y^2 \boxed{-6}x + \boxed{4}y + \boxed{c} = 0$$

Compare with $x^2 + y^2 + \boxed{2g}x + \boxed{2f}y + \boxed{c} = 0$

$$2g = -6,\ 2f = 4,\ c = c$$

$$g = -3, f = 2$$

Centre $= (-g, -f) \Rightarrow$ *Centre* $= (3, -2)$

Centre of the circle is $(3, -2)$ *which lies on* $x + y - 1 = 0$.

$$3 - 2 - 1 = 0 \Rightarrow 0 = 0$$

So the line $x + y - 1 = 0$ *passes through the centre*

$\therefore x + y - 1 = 0$ *is a diameter of the circle for all possible values of* c.

Example 5.4: Find the general equation of the circle whose diameter is the line segment joining the points $(-4, -2)$ and $(1, 1)$

Let $A(-4, -2)$ *and* $B(1, 1)$

$x_1\ y_1 \qquad x_2\ y_2$

$$\boxed{\text{Equation of circle: } (x - x_1)(x - x_2) + (y - y_1)(y - y_2)}$$

$$(x + 4)(x - 1) + (y + 2)(y - 1) = 0$$

$$x^2 - x + 4x - 4 + y^2 - y + 2y - 2 = 0$$

$$x^2 + 3x + y^2 + y - 6 = 0$$

$$x^2 + y^2 + 3x + y - 6 = 0$$

Example 5.5: Examine the position of the point $(2, 3)$ w.r.to the circle $x^2 + y^2 - 6x - 8y + 12 = 0$

Given: $x^2 + y^2 - 6x - 8y + 12 = 0$

$$(2, 3) \Rightarrow x = 2, y = 3$$

$$x^2 + y^2 - 6x - 8y + 12 = 2^2 + 3^2 - 6(2) - 8(3) + 12$$

$$= 4 + 9 - 12 - 24 + 12 = 13 - 24$$

$$= -11 < 0$$

The point $(2, 3)$ *lies inside the circle.*

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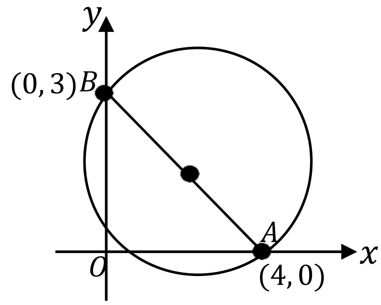
Example 5.6 The line $3x + 4y - 12 = 0$ meets the coordinate axes at A and B. Find the equation of the circle drawn on AB as diameter.

The line $3x + 4y = 12$

$$3x + 4y = 12 \implies \frac{3x}{12} + \frac{4y}{12} = 1$$
$\div 12$

Given line in the intercept form : $\frac{x}{4} + \frac{y}{3} = 1.$

Hence the points A and B are $(4, 0)$ *and* $(0, 3)$.



Equation of the circle in diameter form is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$(x - 4)(x - 0) + (y - 0)(y - 3) = 0$$

$$(x - 4)x + y(y - 3) = 0$$

$$x^2 - 4x + y^2 - 3y = 0 \implies x^2 + y^2 - 4x - 3y = 0$$

Example 5.7: A line $3x + 4y + 10 = 0$ cuts a chord of length 6units on a circle with centre of the circle $(2, 1)$. Find the equation of the circle in general form.

$C(2, 1)$ *is the centre and* $3x + 4y + 10 = 0$ *cuts a chord AB on the circle.*

Lenght of the chord $AB = 6$.

Let M be the midpoint of AB, then $AM = BM = 3$.

$$CM = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right| = \frac{|3(2) + 4(1) + 10|}{\sqrt{3^2 + 4^2}} = \frac{|6 + 4 + 10|}{\sqrt{9 + 16}} = \frac{20}{\sqrt{25}} = \frac{20}{5}$$

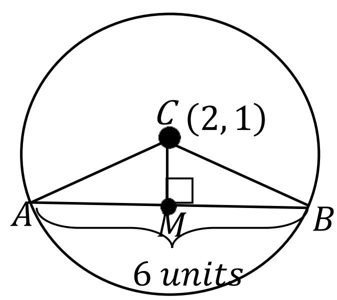
$$CM = 4$$

$$AC^2 = AM^2 + CM^2 \implies AC^2 = 3^2 + 4^2$$

$$AC^2 = 9 + 16 \implies AC^2 = 25$$

$$AC = \sqrt{25} \implies AC = 5$$

radius $= 5$



Centre $(h, k) = (2, 1)$ *and radius* : $r = 5$

Equation of the circle : $(x - h)^2 + (y - k)^2 = r^2$

Equation of the required circle is $(x - 2)^2 + (y - 1)^2 = 5^2$

$$x^2 - 2(2)x + 2^2 + y^2 - 2y + 1^2 = 25$$

$$x^2 - 4x + 4 + y^2 - 2y + 1 - 25 = 0$$

$$x^2 + y^2 - 4x - 2y - 20 = 0$$

Example 5.8: A circle of radius 3 units touches both axes. Find the equation of all possible circles formed in the general form

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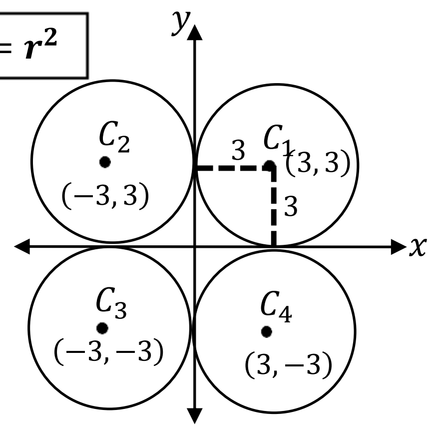
$Centre\ = (\pm 3, \pm 3)$ *and radius*: $r = 3$

Equation of the circle : $(x-h)^2 + (y-k)^2 = r^2$



$$(h,k) = (\pm 3, \pm 3)$$

$$(x \pm 3)^2 + (y \pm 3)^2 = 3^2$$

$$x^2 + 2(\pm 3)x + 3^2 + y^2 + 2(\pm 3)y + 3^2 = 3^2$$

$$x^2 + \pm 6x + 3^2 + y^2 \pm 6y = 0$$

Equation of all possible circles

$$x^2 + y^2 \pm 6x \pm 6y + 9 = 0$$

Example 5.9: ***Find the centre and radius of the circle*** $\mathbf{3x^2 + (a+1)y^2 + 6x - 9y + a + 4 = 0}$

$Co-efficient\ of\ x^2 = Co-efficient\ of\ y^2$

$$3 = a + 1 \Rightarrow a + 1 = 3$$

$$a = 3 - 1 \Rightarrow \boxed{a = 2}$$

$$3x^2 + (2+1)y^2 + 6x - 9y + 2 + 4 = 0$$

$$3x^2 + 3y^2 + 6x - 9y + 6 = 0$$

$\div 3$

$$x^2 + y^2 + \boxed{2}x \boxed{-3}y + \boxed{2} = 0$$

$$Compare\ with\ x^2 + y^2 + \boxed{2g}x + \boxed{2f}y + \boxed{c} = 0$$

$$2g = 2,\ 2f = -3,\ c = 2$$

$$g = 1,\ f = -\frac{3}{2}$$

$$Centre\ = (-g, -f) \Rightarrow Centre\ = \left(-1, \frac{3}{2}\right)$$

$$\boxed{\mathbf{Radius}: r = \sqrt{g^2 + f^2 - c}}$$

$$r = \sqrt{(1)^2 + \left(-\frac{3}{2}\right)^2 - 2} \Rightarrow r = \sqrt{1 + \frac{9}{4} - 2}$$

$$r = \sqrt{\frac{9}{4} - 1} = \sqrt{\frac{9-4}{4}} = \sqrt{\frac{5}{4}} \Rightarrow Radius\ is\ \frac{\sqrt{5}}{2}$$

Example 5.10: ***Find the equation of the circle passing through the points*** $\mathbf{(1,-1), (2,-1)}$ ***and*** $\mathbf{(3,2)}$.

$General\ of\ circle: x^2 + y^2 + 2gx + 2fx + c = 0$

$(1,1)$: $(1)^2 + (1)^2 + 2g(1) + 2f(1) + c = 0$

x, y

$$1 + 1 + 2g + 2f + c = 0 \Rightarrow 2 + 2g + 2f + c = 0$$

$$2g+2f+c=-2 \quad \dots(1)$$

$\underset{x,y}{(2,-1)}$: $(2)^2+(-1)^2+2g(2)+2f(-1)+c=0$

$$4+1+4g-2f+c=0 \implies 5+4g-2g+c=0$$

$$4g-2f+c=-5 \quad \dots(2)$$

$\underset{x,y}{(3,2)}$: $(3)^2+(2)^2+2g(3)+2f(2)+c=0$

$$9+4+6g+4f+c=0 \implies 13+6g+4f+c=0$$

$$6g+4f+c=-13 \quad \dots(3)$$

***Solve* (1) *and* (2)**

$$2g+2f+c=-2$$
$$4g-2f+c=-5$$
$$6g+2c=-7 \dots(4)$$

***Solve* (2) *and* (3)**

$(2)\times 2 \implies 8g-4f+2c=-10$

$(3) \implies 6g+4f+c=-13$

$$14g+3c=-23 \dots(5)$$

***Solve* (4) *and* (5)**

$(4)\times 3 \implies 18g+6c=-21$

$\quad (-)\ (-)\ (+)$

$(5)\times 2 \implies 28g+6c=-46$

$$-10g=25$$

$$\boxed{g=-\frac{5}{2}}$$

$Sub\ g=\frac{-5}{2}\ in\ (4)\ \ 6g+2c=-7$

$$6\left(\frac{-5}{2}\right)+2c=-7 \implies -15+2c=-7$$

$$2c=-7+15 \implies 2c=8$$

$$\boxed{c=4}$$

$Sub\ g=-\frac{5}{2}\ and\ c=4\ in\ (1)\ \ 2g+2f+c=-2$

$$2\left(\frac{-5}{2}\right)+2f+4=-2 \implies -5+2f+4=-2$$

$$2f-1=-2 \implies 2f=-2+1$$

$$2f=-1 \implies \boxed{f=\frac{-1}{2}}$$

$General\ equation:\ x^2+y^2+2gx+2fy+c=0$

$$where\ g=-\frac{5}{2},\ f=\frac{-1}{2}\ and\ c=4$$

$$x^2+y^2+2\left(\frac{-5}{2}\right)x+2\left(\frac{-1}{2}\right)y+4=0$$

$$x^2+y^2-5x-2y+4=0$$

Example 5.11: ***Find the equation of the tangent and normal to the circle*** $x^2 + y^2 = 25$ ***at*** $p(-3, 4)$

Equation of tangent at (x_1, y_1) *in* $xx_1 + yy_1 = 25$

$$(x_1, y_1) = (-3, 4)$$

$$x(-3) + y(4) = 25 \Rightarrow -3x + 4y = 25$$

$$3x - 4y = -25 \Rightarrow 3x - 4y + 25 = 0$$

Equation of normal at (x_1, y_1) ***is*** $yx_1 - xy_1 = 0$

$$(x_1, y_1) = (-3, 4)$$

$$y(-3) - x(4) = 0 \Rightarrow -3y - 4x = 0$$

$$4x + 3y = 0$$

Example 5.12: ***If*** $y = 4x + c$ ***is a tangent to the circle*** $x^2 + y^2 = 9$, ***find c.***

Tangent : $y = 4x + c$ *circle* : $x^2 + y^2 = 9$

compare $y = mx + c$ *compare* : $x^2 + y^2 = a^2$

$m = 4$ $a^2 = 9 \Rightarrow a = 3$

$$c = \pm a\sqrt{(1 + m^2)}$$

$$c = \pm 3\sqrt{(1 + 4^2)} \Rightarrow c = \pm 3\sqrt{(1 + 16)}$$

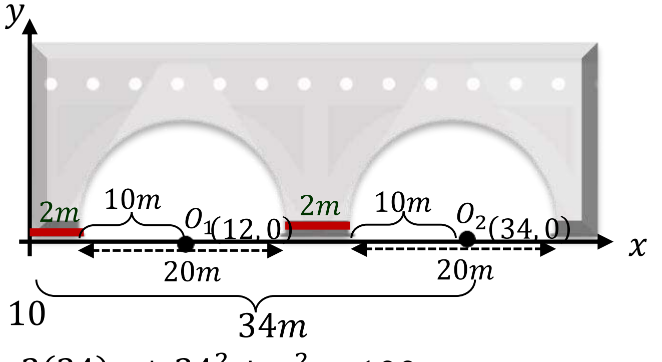
$$c = \pm 3\sqrt{17}$$

Example 5.13: ***A road bridge over an irrigation canal have two semi circular vents each with a span of 20m and the supporting pillars of width 2m. write the equations that model the arches.***

Let O_1, O_2 *be the centres of the two semi circular vents.*

First vent : *centre* O_1 $(12, 0)$ *and radius* $r = 10$

Equation of First vent: $(x - h)^2 + (y - k)^2 = r^2$



$$(x - 12)^2 + (y - 0)^2 = 10^2$$

$$x^2 - 2(12)x + 12^2 + y^2 = 100$$

$$x^2 - 24x + 144 + y^2 - 100 = 0$$

$$x^2 + y^2 - 24x + 44 = 0$$

Second vent :

centre O_2 $(34, 0)$ *and radius* $r = 10$

$$(x - 34)^2 + y^2 = 10^2 \Rightarrow x^2 - 2(34)x + 34^2 + y^2 = 100$$

$$x^2 + y^2 - 68x + 1156 - 100 = 0 \Rightarrow x^2 + y^2 - 68x + 1056 = 0$$

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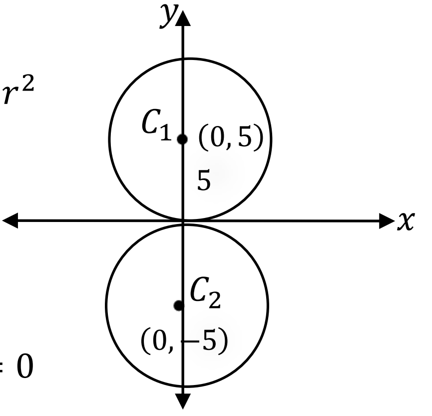
1. *Obtain the equation of the circle with radius 5cm and touching x − axis at the Origin in general form.*



$Centre = (0, \pm 5)$ *and radius*: $r = 5$

Equation of the circle : $(x-h)^2 + (y-k)^2 = r^2$

$(h, k) = (0, \pm 5)$

$(x+0)^2 + (y \pm 5)^2 = 5^2$

$x^2 + y^2 \pm 2(y)(5) + 5^2 = 5^2$

$x^2 + y^2 \pm 10y + 5^2 = 5^2$

Equation of all possible circles $x^2 + y^2 \pm 10y = 0$

2. *Find the equation of the circle with centre* (2, −1) *and passing through the point* (3, 6) *in the standard form*

centre $(h, k) = (2, -1)$

Equation of the circle : $(x-h)^2 + (y-k)^2 = r^2$

$(x+3)^2 + (y+4)^2 = r^2$

It passes through the point $(3, 6)$

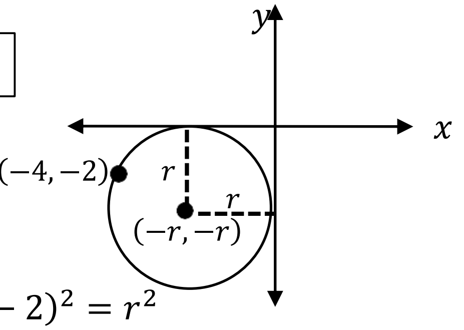
$(3-2)^2 + (6+1)^2 = r^2 \Rightarrow 1^2 + 7^2 = r^2 \Rightarrow 1 + 49 = r^2$

$r^2 = 50$

Required equation: $(x-2)^2 + (y+1)^2 = 50$

3. *Find the equation of circle touches both the axes and pass through the point* (−4, −2) *in general form.*

Equation of circle: $(x-h)^2 + (y-k)^2 = r^2$



Centre: $(h, k) = (-r, -r)$

$(x+r)^2 + (y+r)^2 = r^2 \ldots (1)$

It passes through $(-4, -2)$

$(-4+r)^2 + (-2+r)^2 = r^2 \Rightarrow (r-4)^2 + (r-2)^2 = r^2$

$r^2 - 2(r)(4) + 4^2 + r^2 - 2(r)(2) + 2^2 = r^2$

$r^2 - 8r + 16 + r^2 - 4r + 4 - r^2 = 0$

$r^2 - 12r + 20 = 0 \Rightarrow (r-10)(r-2) = 0$

$r - 10 = 0\ ;\ r - 2 = 0$

$r = 10$	$r = 2$

$(x+r)^2 + (y+r)^2 = r^2$

If $r = 10$: $(x+10)^2 + (y+10)^2 = (10)^2$

$x^2 + 100 + 20x + y^2 + 100 + 20y - 100 = 0$

$x^2 + y^2 + 20x + 20y + 100 = 0$

$If\ r = 2: (x+2)^2 + (y+2)^2 = (2)^2$

$x^2 + 4 + 4x + y^2 + 4 + 4y = 4$

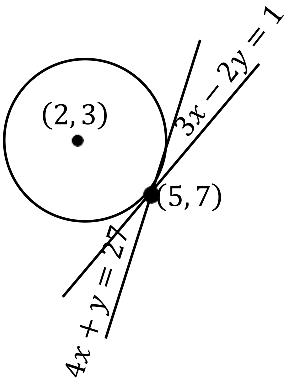
$x^2 + y^2 + 4x + 4y + 4 = 0$

4. *Find the equation of the circle with centre* $(2,3)$ *and passing through the intersection of the lines of* $3x - 2y - 1 = 0$ *and* $4x + y - 27 = 0$

Given : $Centre\ (h,k) = (2,3)$

$3x - 2y = 1 \quad \dots (1)$

$4x + y = 27 \quad \dots (2)$



To find the point of intersection solve (1) *and* (2)

$(1) \Rightarrow \quad 3x - 2y = 1$

$(2) \times 2 \Rightarrow \quad 8x + 2y = 54$

$11x = 55 \quad 5$

$\boxed{x = 5}$

Subs $x = 5$ *in* $(1) 3x - 2y = 1$

$3(5) - 2y = 1$

$15 - 2y = 1 \Rightarrow -2y = 1 - 15 \Rightarrow -2y = -14 \quad 7$

$\boxed{y = 7}$

The point of intersection is $(5, 7)$

$\boxed{Equation\ of\ the\ circle : (x-h)^2 + (y-k)^2 = r^2}$

$(h,k) = (2,3)$

$(x-2)^2 + (y-3)^2 = r^2$

It passes through the point $(5, 7)$

$(5-2)^2 + (7-3)^2 = r^2 \Rightarrow \quad 3^2 + 4^2 = r^2$

$9 + 16 = r^2 \Rightarrow \quad r^2 = 25$

Required equation is $(x-2)^2 + (y-3)^2 = 25$

$x^2 - 2(x)(2) + 2^2 + y^2 - 2(y)(3) + 3^2 = 25$

$x^2 - 4x + 4 + y^2 - 6y + 9 = 25 \Rightarrow x^2 + y^2 - 4x - 6y + 13 - 25 = 0$

$x^2 + y^2 - 4x - 6y - 12 = 0$

5. *Obtain equation of the circle for which* $(3, 4)$ *and* $(2, -7)$ *are the ends of a diameter.*

Let $A(\underset{x_1\ y_1}{3, 4})$ *and* $B\ (\underset{x_2\ y_2}{2, -7})$

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$Equation\ of\ circle: (x - x_1)(x - x_2) + (y - y_1)(y - y_2)$

$$(x-3)(x-2) + (y-4)(y+7) = 0$$

$$x^2 - 2x - 3x + 6 + y^2 + 7y - 4y - 28 = 0$$

$$x^2 - 5x + y^2 + 3y - 22 = 0$$

$$x^2 + y^2 - 5x + 3y - 22 = 0$$

6. *Find the equation of the circle throught the points* $(1,0), (-1,0)$ *and* $(0,1)$

$General\ of\ circle: x^2 + y^2 + 2gx + 2fy + c = 0$

$(1,0)$ [x, y]: $1^2 + 0^2 + 2g(1) + 2f(0) + c = 0 \Rightarrow 1 + 2g + c = 0$

$$2g + c = -1 \quad \dots(1)$$

$(-1,0)$ [x, y]: $(-1)^2 + 0^2 + 2g(-1) + 2f(0) + c = 0 \Rightarrow 1 - 2g + c = 0$

$$-2g + c = -1 \quad \dots(2)$$

$(0,1)$ [x, y]: $0^2 + 1^2 + 2g(0) + 2f(1) + c = 0 \Rightarrow 1 + 2f + c = 0$

$$2f + c = -1 \dots(3)$$

$Solve\ (1)\ and\ (2)$

$(1) \Rightarrow 2g + c = -1$

$(2) \Rightarrow -2g + c = -1$

$$2c = -2$$

$$c = -\frac{2}{2} \Rightarrow \boxed{c = -1}$$

$Sub\ c = -1\ in\ (1)\ 2g + c = -1$

$$2g - 1 = -1 \Rightarrow 2g = -1 + 1 \Rightarrow 2g = 0 \Rightarrow \boxed{g = 0}$$

$Sub\ c = -1\ in\ (3)\ 2f + c = -1$

$$2f - 1 = -1 \Rightarrow 2f = -1 + 1 \Rightarrow 2f = 0 \Rightarrow \boxed{f = 0}$$

$g = 0, f = 0, c = -1\ \ in\ x^2 + y^2 + 2gx + 2fy + c = 0$

$$x^2 + y^2 + 2(0)x + 2(0)y - 1 = 0 \Rightarrow x^2 + y^2 - 1 = 0$$

$$\boxed{x^2 + y^2 = 1}$$

7. *A circle of area* 9π *square unit has two of its diameter along the lines* $x + y = 5$ *and* $x - y = 1$. *Find the equation of the circle.*

$Area\ of\ a\ circle = 9\pi$

$$\pi r^2 = 9\pi \Rightarrow r^2 = 9 \Rightarrow r = \sqrt{9} \Rightarrow r = 3$$

$Point\ of\ intersection\ of\ two\ line = centre\ of\ the\ circle$

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$x+y=5 \ldots(1)$ and $x-y=1 \ldots(2)$

$(1) \Rightarrow x+y=5$

$(2) \Rightarrow x-y=1$

$2x=6$ 3

$x=3$

Subs $x=3$ in (1) $x+y=5$

$3+y=5 \Rightarrow y=5-3 \Rightarrow y=2$

Centre is $(3,2)$

Equation of the circle: $(x-h)^2+(y-k)^2=r^2$

Centre $(h,k)=(3,2)$ and $r=3$

$$(x-3)^2+(y-2)^2=3^2$$

$$x^2-2(3)x+3^2+y^2-2(2)y+2^2=9$$

$$x^2-6x+9+y^2-4y+4=9$$

$$x^2+y^2-6x-4y+4=0$$

Ex: 8. If $y=2\sqrt{2}x+c$ is a tangent to the circle $x^2+y^2=16$, find c.

Tangent : $y=2\sqrt{2}x+c$ circle : $x^2+y^2=16$

compare $y=mx+c$ compare : $x^2+y^2=a^2$

$m=2\sqrt{2}$ $a^2=16 \Rightarrow a=4$

$$c=\pm a\sqrt{(1+m^2)}$$

$$c=\pm 4\sqrt{\left[1+\left(2\sqrt{2}\right)^2\right]} \Rightarrow c=\pm 4\sqrt{(1+8)}$$

$$c=\pm 4\sqrt{9} \Rightarrow c=\pm 4(3)$$

$$c=\pm 12$$

9. Find the equation of the tangent and normal to the circle $x^2+y^2-6x+6y-8=0$ at $(2,2)$

Equation of the tangent at (x_1,y_1) is

$$xx_1+yy_1+2g\left(\frac{x+x_1}{2}\right)+2f\left(\frac{y+y_1}{2}\right)+c=0$$

$$xx_1+yy_1-6\left(\frac{x+x_1}{2}\right)+6\left(\frac{y+y_1}{2}\right)-8=0$$

$$xx_1+yy_1-3(x+x_1)+3(y+y_1)-8=0$$

$$(x_1,y_1)=(2,2)$$

$$x(2)+y(2)-3(x+2)+3(y+2)-8=0$$

$$2x+2y-3x-6+3y+6-8=0$$

$$2x + 2y - 3x + 3y - 8 = 0 \implies -x + 5y - 8 = 0$$

$$x - 5y + 8 = 0$$

$$Equation\ of\ normal\ form : 5x + y + k = 0$$

$$(x, y) = (2, 2)$$

$$5(2) + 2 + k = 0 \implies 12 + k = 0 \implies k = -12$$

$$Equation\ of\ normal : 5x + y - 12 = 0$$

$$y(2) - x(2) - 6\left(\frac{y-2}{2}\right) - 6\left(\frac{x-2}{2}\right) = 0$$

$$2y - 2x - 3(y-2) - 3(x-2) = 0$$

$$2y - 2x - 3y + 6 - 3x + 6 = 0$$

$$\boxed{x + 5y - 12 = 0}$$

10. Determine whether the points $(-2, 1)$ $(0, 0)$ and $(-4, -3)$ lies outside, on or inside the circle $x^2 + y^2 - 5x + 2y - 5 = 0$

$(-2, 1) \implies x = -2, y = 1$

$$x^2 + y^2 - 5x + 2y - 5 = (-2)^2 + 1^2 - 5(-2) + 2(1) - 5$$

$$= 4 + 1 + 10 + 2 - 5 = 17 - 5 = 12 > 0$$

The point $(-2, 1)$ lies outside the circle.

$(0, 0) \implies x = 0, y = 0$

$$x^2 + y^2 - 5x + 2y - 5 = (0)^2 + (0)^2 - 5(0) + 2(0) - 5$$

$$= 0 - 5 = -5 < 0$$

The point $(0, 0)$ lies inside the circle.

$(-4, -3) \implies x = -4, y = -3$

$$x^2 + y^2 - 5x + 2y - 5 = (-4)^2 + (-3)^2 - 5(-4) + 2(-3) - 5$$

$$= 16 + 9 + 20 - 6 - 5$$

$$= 45 - 11 = 34 > 0$$

The point $(-4, -3)$ lies outside the circle.

11. Find centre and radius of the following circles.

i) $x^2 + (y + 2)^2 = 0$ ii) $x^2 + y^2 + 6x - 4y + 4 = 0$

iii) $x^2 + y^2 - x + 2y - 3 = 0$ iv) $2x^2 + 2y^2 - 6x + 4y + 2 = 0$

i) $x^2 + (y + 2)^2 = 0$

$$(x - 0)^2 + (y + 2)^2 = 0$$

Compare $(x - h)^2 + (y - k)^2 = r^2$

$$h = 0,\ k = -2, r^2 = 0 \implies r = 0$$

Centre $= (0, -2)$ *and Radius is* 0

ii) $x^2 + y^2 + \boxed{6}x \boxed{-4}y + \boxed{4} = 0$

Compare with $x^2 + y^2 + \boxed{2g}x + \boxed{2f}y + \boxed{c} = 0$

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$$2g = 6,\ 2f = -4,\ c = 4$$

$$g = 3,\ f = -2$$

$$Centre = (-g, -f) \implies Centre = (-3, 2)$$

$$Radius: r = \sqrt{g^2 + f^2 - c}$$

$$r = \sqrt{(3)^2 + (-2)^2 - 4} \implies r = \sqrt{9 + 4 - 4} \implies r = \sqrt{9}$$

$Radius\ is\ 3$

$iii)\ x^2 + y^2 - x + 2y - 3 = 0$

$$Compare\ with\ x^2 + y^2 + 2gx + 2fy + c = 0$$

$$2g = -1,\ 2f = 2,\ c = -3 \implies g = -\frac{1}{2}, f = 1$$

$$Centre = (-g, -f) \implies Centre = \left(\frac{1}{2}, -1\right)$$

$$Radius: r = \sqrt{g^2 + f^2 - c}$$

$$r = \sqrt{\left(-\frac{1}{2}\right)^2 + 1^2 + 3} \implies r = \sqrt{\frac{1}{4} + 1 + 3} \implies r = \sqrt{\frac{1}{4} + 4}$$

$$r = \sqrt{\frac{1 + 16}{4}} \implies r = \sqrt{\frac{17}{4}} \quad \therefore Radius\ is\ \frac{\sqrt{17}}{2}$$

$iv)\ 2x^2 + 2y^2 - 6x + 4y + 2 = 0$

$\div 2$

$$x^2 + y^2 - 3x + 2y + 1 = 0$$

$$Compare\ with\ x^2 + y^2 + 2gx + 2fy + c = 0$$

$$2g = -3,\ 2f = 2,\ c = 1$$

$$g = -\frac{3}{2}, f = 1$$

$$Centre = (-g, -f) \implies Centre = \left(\frac{3}{2}, -1\right)$$

$$Radius: r = \sqrt{g^2 + f^2 - c}$$

$$r = \sqrt{\left(-\frac{3}{2}\right)^2 + 1^2 - 1} \implies r = \sqrt{\frac{9}{4} + 1 - 1} \implies r = \sqrt{\frac{9}{4}}$$

$$\therefore Radius\ is\ \frac{3}{2}$$

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12. If the equation $3x^2+(3-p)xy+qy^2-2px=8pq$ represents a cicle find p and q. Also determine the centre, radius of the circle.

$$3x^2+(3-p)xy+qy^2-2px=8pq$$

$$Co-efficient\ of\ xy=0$$

$$3-p=0$$

$$-p=-3 \Rightarrow \boxed{p=3}$$

$$3x^2+(3-3)xy+qy^2-2(3)x=8(3)q$$

$$3x^2+qy^2-6x=24q$$

$$3x^2+qy^2-6x-24q=0$$

$$Co-efficient\ of\ x^2=Co-efficient\ of\ y^2$$

$$3=q \Rightarrow \boxed{q=3}$$

$$3x^2+3y^2-6x-24(3)=0$$

$$\div 3$$

$$x^2+y^2-2x-24=0$$

$$x^2+y^2-2x+0y-24=0$$

$$Compare\ with\ x^2+y^2+2gx+2fy+c=0$$

$$2g=-2,\ 2f=0,\ c=-24$$

$$-1$$

$$g=-1,\ f=0$$

$$Centre=(-g,-f) \Rightarrow Centre=(1,0)$$

$$Radius: r=\sqrt{g^2+f^2-c} \Rightarrow r=\sqrt{(-1)^2+(0)^2+24}$$

$$r=\sqrt{1+24} \Rightarrow r=\sqrt{25}$$

$$Radius\ is\ 5$$

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EXERCISE : 5.2

Example 5.14. Find the equation of the parabola with focus $(-\sqrt{2},0)$, $x=\sqrt{2}$

Distance between vertex and focus $VF = a = \sqrt{2}$

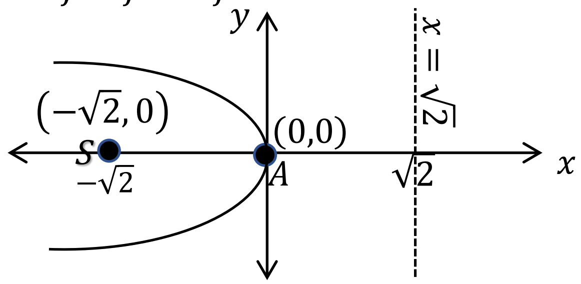
The equation of the parabola is open left of the form:

$(y-k)^2 = -4a(x-h)$

$V(\underset{h}{0},\underset{k}{0})$ and $a=\sqrt{2}$

$(y-0)^2 = -4(\sqrt{2})(x-0)$

$y^2 = -4\sqrt{2}x$



Example 5.15 Find the equation of the parabola if vertex $(5,-2)$ and focus$(2,-2)$

Distance between vertex and focus $VS = a$

$V(\underset{x_1}{5},\underset{y_1}{-2})$ and $F(\underset{x_2}{2},\underset{y_2}{-2})$

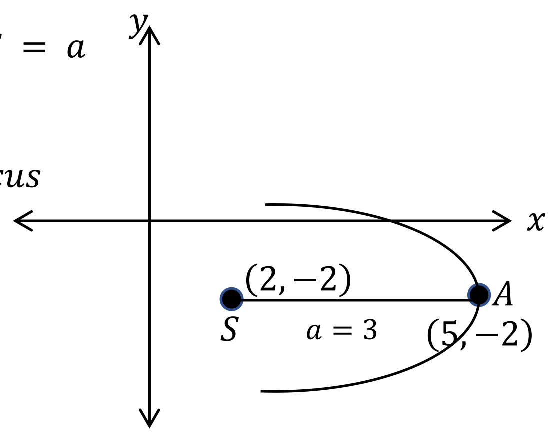
$VF =$ Distance between vertex and focus

$a = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$a = \sqrt{(2-5)^2+(-2+2)^2}$

$a = \sqrt{(-3)^2+(0)^2} \implies a = \sqrt{9}$

$\boxed{a=3}$



The equation of the parabola is open lef of the form:

$(y-k)^2 = -4a(x-h)$

$V(\underset{h}{5},\underset{k}{-2})$ and $a=3$

$(y+2)^2 = -4(3)(x-5) \implies (y+2)^2 = -12(x-5)$

Example 5.16 Find the equation of the parabola with vertex $(-1,-2)$ axis parallel to $y-$axis and passing through the point $(3,6)$

The equation of the parabola is open upward of the form:

$\boxed{(x-h)^2 = 4a(y-k)}$

The vertex is $V(\underset{h}{-1},\underset{k}{-2})$

$(x+1)^2 = 4a(y+2) \ldots (1)$

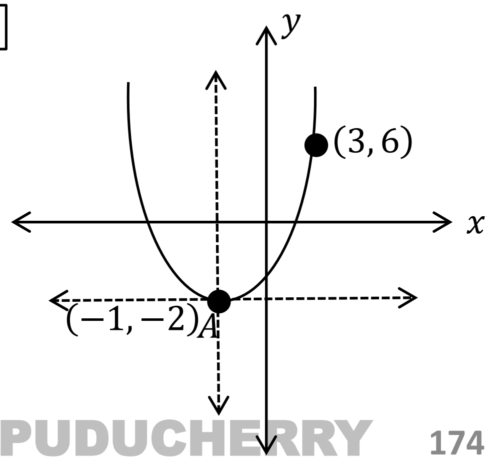
It passes through the point $(\underset{x,y}{3,6})$

$(3+1)^2 = 4a(6+2)$

$4^2 = 4a(8)$

$\overset{2}{\cancel{32}}a = \cancel{16} \implies 2a = 1 \implies a = \frac{1}{2}$

sub $a=\frac{1}{2}$ in(1) $(x+1)^2 = 4a(y+2)$



$$(x+1)^2 = 4\left(\frac{1}{2}\right)(y+2) \implies (x+1)^2 = 2(y+2)$$

***Example* 5.17 Find the vertex, focus, directrix, length of the latus rectum for the parabolas $x^2 - 4x - 5y - 1 = 0$**

$$x^2 - 4x - 5y - 1 = 0$$

$$x^2 - 4x = 5y + 1 \qquad \boxed{\frac{4}{2} = 2}$$

$$\underbrace{x^2 - 4x + 2^2} - 2^2 = 5y + 1$$

$$(x-2)^2 - 4 = 5y + 1 \implies (x-2)^2 = 5y + 1 + 4$$

$$(x-2)^2 = 5y + 5 \implies (x-2)^2 = 5(y+1)$$

$$\boxed{Compare\ with\ (x-h)^2 = 4a(y-k)} \quad (open\ Upward)$$

$$h = 2, k = -1, 4a = 5$$

$$\boxed{a = \frac{5}{4}}$$

$Vertex\ A\ (h,k)\ is\ (2,-1)$

$$focus\ (0+h, a+k) = \left(0+2, \frac{5}{4} - 1\right) = \left(2, \frac{5-4}{4}\right)$$

$$= \left(2, \frac{1}{4}\right)$$

$\boxed{Equation\ of\ directrix}$ $\boxed{y = -a + k}$

$$y = -a + k \implies y = -\frac{5}{4} - 1$$

$$y = \frac{-5-4}{4} \implies y = -\frac{9}{4}$$

$$\boxed{Length\ of\ Latus\ rectum:\ 4a = 4\left(\frac{5}{4}\right) = 5}$$

***Example* 5.18 *Find the equation of the ellipse if the foci are* $(\pm 2, 0)$ *and the vertices are* $(\pm 3, 0)$**

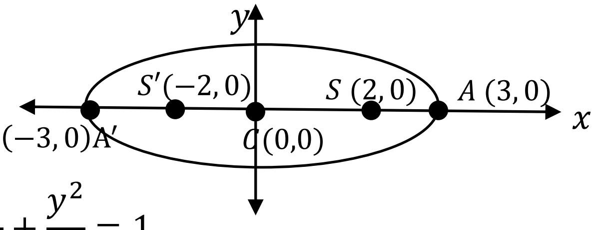
The foci are $S(2,0)$ and $S'(-2,0)$ vertices are $A(3,0)$ and $A'(-3,0)$

$$c = 2\ and\ a = 3$$

$$a^2 = 9$$

$$b^2 = a^2 - c^2 \implies b^2 = 3^2 - 2^2$$

$$b^2 = 9 - 4 \implies b^2 = 5$$



$$\therefore The\ equation\ of\ the\ ellipse\ is\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{9}+\frac{y^2}{5}=1$$

Example 5.19 Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, one of the foci is $(2,3)$, and a directrix is $x=7$. Also find the lenght of the major and minor axes of the ellipse

Given : $F(2,3)$; *directrix* : $x-7=0$

Let $P(x,y)$ *be any point on the parabola*

Definition : $\frac{SP}{PM}=e \Longrightarrow \boxed{FP=ePM}$

$F(2,3)$ (x_1 y_1) *and* $P(x,y)$ (x_2 y_2)

$$\boxed{FP=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} \Longrightarrow FP=\sqrt{(x-2)^2+(y-3)^2}$$

$$\boxed{PM=\pm\frac{lx+my+n}{\sqrt{l^2+m^2}}} \Longrightarrow PM=\pm\frac{x-7}{\sqrt{1^2+0^2}} \Longrightarrow \boxed{PM=\pm(x-7)}$$

$$\boxed{FP=ePM}$$

$$\sqrt{(x-2)^2+(y-3)^2}=\pm\frac{1}{2}(x-7)$$

Squaring on both sides

$$(x-2)^2+(y-3)^2=\left(\frac{1}{2}\right)^2(x-7)^2$$

$$x^2-2(x)(2)+2^2+y^2-2(y)(3)+3^2=\frac{1}{4}(x^2-2\times x\times 7+7^2)$$

$$x^2-4x+4+y^2-6y+9=\frac{1}{4}(x^2-14x+49)$$

$$x^2-4x+y^2-6y+13=\frac{1}{4}(x^2-14x+49)$$

$$4[x^2-4x+y^2-6y+13]=x^2-14x+49$$

$$4x^2-16x+4y^2-24y+52-x^2+14x-49=0$$

$$3x^2-2x+4y^2-24y+3=0$$

$$\boxed{\frac{\frac{2}{3}}{2}=\frac{1}{3}}$$

$$3\left(x^2-\frac{2}{3}x\right)+4(y^2-6y)+3=0$$

$$\boxed{\frac{6}{2}=3}$$

$$3\left[\underbrace{x^2-\frac{2}{3}x+\left(\frac{1}{3}\right)^2}-\left(\frac{1}{3}\right)^2\right]+4(\underbrace{y^2-6y+3^2}-3^2)+3=0$$

$$3\left[\left(x-\frac{1}{3}\right)^2-\frac{1}{9}\right]+4[(y-3)^2-9]+3=0$$

$$3\left(x-\frac{1}{3}\right)^2-\frac{1}{3}+4(y-3)^2-36+3=0$$

$$3\left(x-\frac{1}{3}\right)^2-\frac{1}{3}+4(y-3)^2-33=0 \Rightarrow 3\left(x-\frac{1}{3}\right)^2+4(y-3)^2=\frac{1}{3}+33$$

$$3\left(x-\frac{1}{3}\right)^2+4(y-3)^2=\frac{1}{3}+33 \Rightarrow 3\left(x-\frac{1}{3}\right)^2+4(y-3)^2=\frac{1+99}{3}$$

$$3\left(x-\frac{1}{3}\right)^2+4(y-3)^2=\frac{100}{3} \Rightarrow \frac{3\left(x-\frac{1}{3}\right)^2}{\frac{100}{3}}+\frac{4(y-3)^2}{\frac{100}{3}}=1$$

$$\div\frac{100}{3}$$

$$\frac{\left(x-\frac{1}{3}\right)^2}{\frac{100}{3}\times\frac{1}{3}}+\frac{(y-3)^2}{\frac{100}{3}\times\frac{1}{4}}=1 \Rightarrow \frac{\left(x-\frac{1}{3}\right)^2}{\frac{100}{3}\times\frac{1}{3}}+\frac{(y-3)^2}{\frac{100}{3}\times\frac{1}{4}}=1$$

$$\frac{\left(x-\frac{1}{3}\right)^2}{\frac{100}{9}}+\frac{(y-3)^2}{\frac{100}{12}}=1$$

compare with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$a^2=\frac{100}{9} \Rightarrow a=\sqrt{\frac{100}{9}} \Rightarrow a=\frac{10}{3}$$

$$b^2=\frac{100}{9} \Rightarrow b=\sqrt{\frac{100}{12}} \Rightarrow b=\frac{10}{2\sqrt{3}}$$

Length of Major axis $=2a \Rightarrow 2a=\frac{20}{3}$

Length of Minor axis $=2b \Rightarrow 2b=2\times\frac{10}{2\sqrt{3}}=\frac{10}{\sqrt{3}}$

Example 5.20 Find the foci, vertices and length of major and minor axis of the conic $\mathbf{4x^2+36y^2+40x-288y+532=0}$

$$4x^2+36y^2+40x-288y+532=0 \qquad \boxed{\frac{10}{2}=5} \quad \boxed{\frac{8}{2}=4}$$

$$4x^2+40x+36y^2-288y+532=0$$

$$4(x^2+10x)+36(y^2-8y)+532=0$$

$$4(\underbrace{x^2+10x+5^2}-5^2)+36(\underbrace{y^2-8y+4^2}-4^2)+532=0$$

$$4[(x+5)^2-25]+36[(y-4)^2-16]+532=0$$

$$4(x+5)^2-100+36(y-4)^2-576+532=0$$

$$4(x+5)^2+36(y-4)^2-144=0$$

$$4(x+5)^2+36(y-4)^2=144$$

$$\div 144$$

$$\frac{(x+5)^2}{36}+\frac{(y-4)^2}{4}=1$$

Compare with $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ *(major axis is along x− axis)*

$$h=-5, k=4$$

$$a^2=36,\ b^2=4 \Longrightarrow a=6,\ b=2$$

$\boxed{The\ length\ of\ major\ axis\ =2a}=2(6)=12$

$\boxed{The\ length\ of\ minor\ axis\ =2b}=2(2)=4$

$$vertices\ (\pm a+h, 0+k)=(\pm 6-5, 0+4)=(6-5, 4), (-6-5, 4)$$

$$=(1,4), (-11,4)$$

$$c^2=a^2-b^2 \Longrightarrow c^2=36-4$$

$$c^2=32 \Longrightarrow c=\sqrt{32} \Longrightarrow c=\sqrt{16\times 2} \Longrightarrow \boxed{c=\pm 4\sqrt{2}}$$

$$focus\ (\pm c+h, 0+k)=\left(\pm 4\sqrt{2}-5, 4\right)=\left(4\sqrt{2}-5, 4\right), \left(-4\sqrt{2}-5, 4\right)$$

***Example 5.21**: For the ellipse $4x^2+y^2+24x-2y+21=0$, find the centre, vertices, and foci. Also prove that the length of latus rectum is 2*

$$4x^2+y^2+24x-2y+21=0$$

$$4x^2+24x+y^2-2y+21=0$$

$\boxed{\frac{6}{2}=3}$ $\boxed{\frac{2}{2}=1}$

$$4(x^2+6x)+(y^2-2y)+21=0$$

$$4(x^2+6x+3^2-3^2)+(y^2-2y+1^2-1^2)+21=0$$

$$4[(x+3)^2-9]+[(y-1)^2-1]+21=0$$

$$4(x+3)^2-36+(y-1)^2-1+21=0$$

$$4(x+3)^2+(y-1)^2-16=0$$

$$4(x+3)^2+(y-1)^2=16$$

$$\div 16$$

$$\frac{4(x+3)^2}{16}+\frac{(y-1)^2}{16}=1 \Longrightarrow \frac{(x+3)^2}{4}+\frac{(y-1)^2}{16}=1$$

Compare with $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$

(major axis is along Y − axis)

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$$h = -3, k = 1$$

$$b^2 = 4,\ a^2 = 16 \Longrightarrow \quad \mathrm{b} = \sqrt{4},\ \ \mathrm{a} = \sqrt{16}$$

$$b = 2, a = 4$$

$$centre\ (h, k) \Longrightarrow (-3, 1)$$

$$vertices\ (0 + h, \pm a + k) = (-3, \pm 4 + 1) = (3, 4 + 1), (3, -4 + 1)$$

$$= (3, 5), (3, -3)$$

$$c^2 = a^2 - b^2 \ \Longrightarrow \ c^2 = 16 - 4$$

$$c^2 = 12 \Longrightarrow c = \sqrt{12} \Longrightarrow c = \sqrt{4 \times 3} \Longrightarrow \boxed{c = \pm 2\sqrt{3}}$$

$$focus\ (0 + h, \pm c + k) = \left(-3, \pm 2\sqrt{3} + 1\right) \ = \left(3, 2\sqrt{3} + 1\right), \left(3, -2\sqrt{3} + 1\right)$$

$$\boxed{\text{Length of latus rectum} = \frac{2b^2}{a}} \quad \text{Where } \mathrm{b}^2 = 4, \mathrm{a} = 4$$

$$= \frac{2(4)}{4} = 2$$

Example 5.22 Find the equation of the hyperbola if the are vertices $(\mathbf{0}, \pm\mathbf{4})$ and the foci are $(\mathbf{0}, \pm\mathbf{6})$

The foci are $S(0, 6)$ and $S'(0, -6)$. vertices are $\mathrm{A}(0, 4)$ and $\mathrm{A}'(0, -4)$

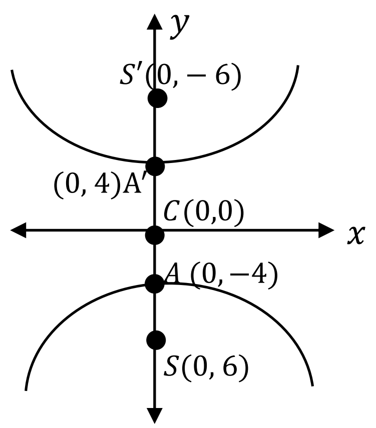
$$c = 6 \ and\ a = 4$$

$$a^2 = 16$$

$$\boxed{c^2 = a^2 + b^2}$$

$$b^2 = c^2 - a^2 \Longrightarrow \ b^2 = 6^2 - 4^2$$

$$b^2 = 36 - 16 \Longrightarrow b^2 = 20$$



$$\boxed{\therefore \ The\ equation\ of\ the\ hyperbola\ is\ \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1}$$

$$\boxed{\frac{y^2}{16} - \frac{x^2}{20} = 1}$$

Example 5.23 Find the vertices, foci for the hyperbola $\mathbf{9x^2 - 16y^2 = 144}$

$$9x^2 - 16y^2 = 144$$

$$\div 144$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Compare with $\frac{x^2}{\mathrm{a}^2} - \frac{y^2}{\mathrm{b}^2} = 1$ *(transverse axis is along X− axis)*

$$\mathrm{a}^2 = 16, b^2 = 9 \Longrightarrow \ a = 4,\ b = 3$$

$$c^2 = a^2 + b^2 \ \ \Longrightarrow c = \sqrt{a^2 + b^2}$$

$$c = \sqrt{16 + 9} \ \ \Longrightarrow c = \sqrt{25}$$

$$\boxed{c = 5}$$

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Centre: $(0,0)$

Foci: $(\pm c, 0) \Rightarrow (\pm 5, 0) \Rightarrow (5,0), (-5,0)$

Vertices: $(\pm a, 0) \Rightarrow (\pm 4, 0) \Rightarrow (4,0), (-4,0)$

Example 5.24 Find the centre, foci, and eccentricity of the hyperbola

$$\mathbf{11x^2 - 25y^2 - 44x + 50y - 256 = 0}$$

$$11x^2 - 44x - 25y^2 + 50y - 256 = 0$$

$$11(x^2 - 4x) - 25(y^2 - 2y) - 256 = 0$$

$$11(x^2 - 4x + 2^2 - 2^2) - 25(y^2 - 2y + 1^2 - 1^2) - 256 = 0$$

$$11[(x-2)^2 - 4] - 25[(y-1)^2 - 1] - 256 = 0$$

$$11(x-2)^2 - 44 - 25(y-1)^2 + 25 - 256 = 0$$

$$11(x-2)^2 - 25(y-1)^2 - 275 = 0$$

$$11(x-2)^2 - 25(y-1)^2 - 275 = 0 \Rightarrow 11(x-2)^2 - 25(y-1)^2 = 275$$

$\div 275$

$$\frac{(x-2)^2}{25} - \frac{(y-1)^2}{11} = 1$$

$$compare\ with\ \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \ \therefore [transerve\ axis\ along\ x - axis]$$

$h = 2$, $k = 1$

$a^2 = 25, b^2 = 11 \Rightarrow a = 5, b = \sqrt{11}$

$centre : (h,k) = (2,1)$

$c^2 = a^2 + b^2 \Rightarrow c = \sqrt{a^2 + b^2}$

$c = \sqrt{25 + 11} \Rightarrow c = \sqrt{36} \Rightarrow \boxed{c = 6}$

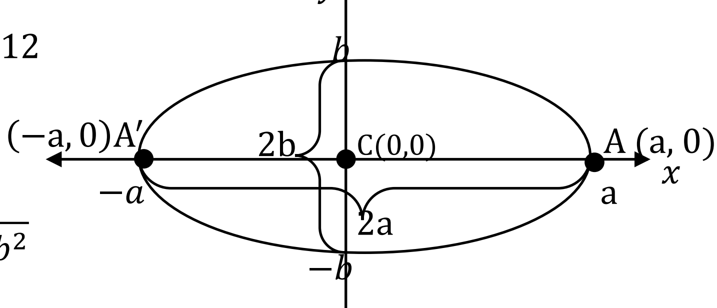
$foci : (\pm c + h, 0 + k) \Rightarrow (\pm 6 + 2, 0 + 1)$

$= (6 + 2, 1), (-6 + 2, 1) = (8,1), (-4, 1)$

To Find eccentricity (e)

$$e = \frac{c}{a} \Rightarrow \boxed{e = \frac{6}{5}}$$

Example 5.25 The orbit of Halley's Comet (Fig. 5.51) is an ellipse 36.18 astronomical units long and by 9.12 astronomical units wide. Find its eccentricity



$Given\ that\ 2a = 36.18, 2b = 9.12$

$$a = \frac{36.18}{2}, b = \frac{9.12}{2}$$

$a = 18.09, b = 4.56$

$c^2 = a^2 - b^2 \Rightarrow c = \sqrt{a^2 - b^2}$

$c = \sqrt{(18.09)^2 - (4.56)^2}$

$$c = \sqrt{327.2481 - 20.7936} \Rightarrow c = \sqrt{306.4545}$$

$$\boxed{c = 17.506}$$

$\boxed{To\ Find\ eccentricity\ (e)}$

$$e = \frac{c}{a} \Rightarrow e = \frac{17.506}{18.09} \Rightarrow \boxed{e = 0.97}$$

Ex: 1(i). Find the equation of the parabola with focus(4, 0), x = −4

$\boxed{Distance\ between\ vertex\ and\ focus\ VS = a = 4}$

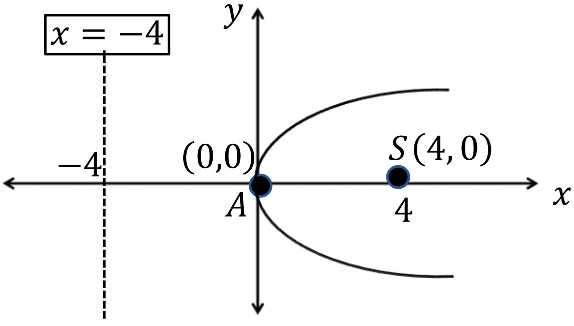
$\boxed{The\ equation\ of\ the\ parabola\ is\ open\ right\ of\ the\ form:}$

$$(y-k)^2 = 4a(x-h)$$

$$V(\underset{h}{0}, \underset{k}{0})\ and\ a = 4$$

$$(y-0)^2 = 4(4)(x-0)$$

$$\boxed{y^2 = 16x}$$



Ex: 1(ii) Find the equation of the parabola which passes through the point (2, −3) and Symmetric about y − axis

The equation of the parabola is open downward of the form:

$$\boxed{x^2 = -4ay} \dots (1)$$

It passes through the point $(\underset{x}{2}, \underset{y}{-3})$

$$2^2 = -4a(-3)$$

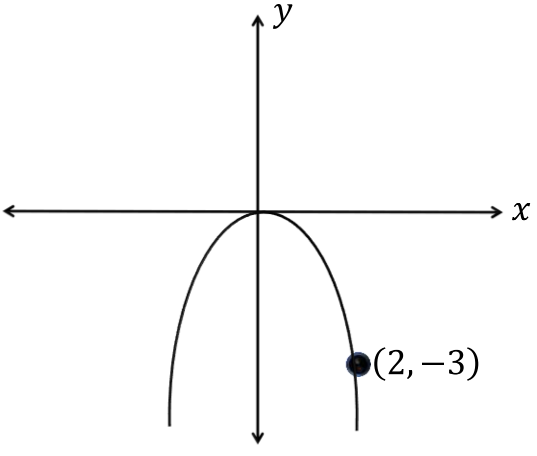
$$4 = 12a \Rightarrow \frac{4}{12_{3}} = a$$

$$\boxed{a = \frac{1}{3}}$$

$$sub\ a = \frac{1}{3}\ in\ (1)\ x^2 = -4ay$$

$$x^2 = -4\left(\frac{1}{3}\right)y \Rightarrow x^2 = -\frac{4}{3}y$$

$$\boxed{3x^2 = -4y}$$



Ex: 1(iii) Find the equation of the parabola if vertex (1, −2) and focus(4, −2)

Distance between vertex and focus VF = a

$$V(\underset{x_1}{1}, \underset{y_1}{-2})\ and\ F(\underset{x_2}{4}, \underset{y_2}{-2})$$

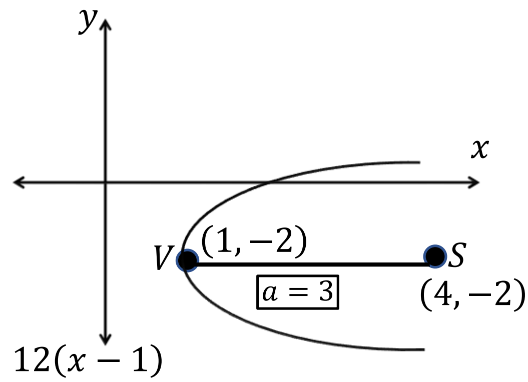
$$VF = Distance\ between\ vertex\ and\ focus$$

$$\boxed{a = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}$$

$a = \sqrt{(4-1)^2 + (-2+2)^2}$

$a = \sqrt{(3)^2 + (0)^2} \implies a = \sqrt{9}$

$\boxed{a = 3}$



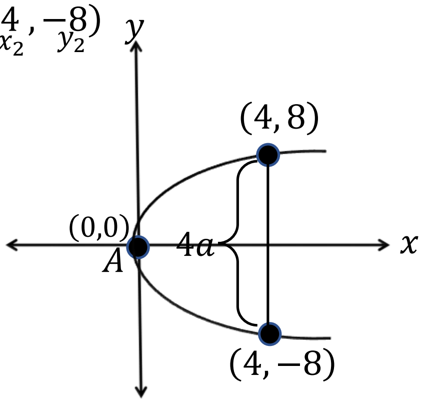
The equation of the parabola is open lef of the form: $(y-k)^2 = 4a(x-h)$

$V(\underset{h}{1}, \underset{k}{-2})$ and $a = 3$

$(y+2)^2 = 4(3)(x-1) \implies (y+2)^2 = 12(x-1)$

Ex: 1(iv) Find the equation of the parabola if end points of latus rectum $(4,-8)$ and $(4,8)$

Latus Rectum = Distance between $(\underset{x_1}{4}, \underset{y_1}{8})$ *and* $(\underset{x_2}{4}, \underset{y_2}{-8})$



$\boxed{4a = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}$

$4a = \sqrt{(4-4)^2 + (-8-8)^2}$

$4a = \sqrt{(0)^2 + (-16)^2} \implies 4a = \sqrt{16^2}$

$4a = 16 \implies \boxed{a = 4}$

The equation of the parabola is open right of the form: $(y-k)^2 = 4a(x-h)$

$A(\underset{h}{0}, \underset{k}{0})$ and $a = 4$

$(y-0)^2 = 4(4)(x-0)$

$\boxed{y^2 = 16x}$

2 (i) Find the equation of the ellipse if the foci are $(\pm 3, 0)$ and $e = \frac{1}{2}$

From the given data the major axis is along x − axis.

$c = 3$ and $e = \frac{1}{2}$

$ae = 3 \implies a \times \frac{1}{2} = 3 \implies a = 3 \times 2$

$a = 6 \implies a^2 = 36$

$c = 3 \implies c^2 = 9$

$b^2 = a^2 - c^2 \implies b^2 = 36 - 9$

$b^2 = 27$

$\therefore$ *The equation of the ellipse is* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{36} + \frac{y^2}{27} = 1$$

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2 (ii). Find the equation of the ellipse if the foci are $(0, \pm 4)$ and endpoints of major axis are $(0, \pm 5)$

From the given data the major axis is along $y-$axis.

$$c = 4 \text{ and } a = 5$$
$$c = 4 \Rightarrow c^2 = 16$$
$$a = 5 \Rightarrow a^2 = 25$$
$$b^2 = a^2 - c^2 \Rightarrow b^2 = 25 - 16$$
$$b^2 = 9$$

$\therefore$ The equation of the ellipse is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

2(iii). Find the equation of the ellipse if the lenght of latus rectum is 8 and eccentricity is 3/5 and the major axis on $x-$axis

The major axis is $x-$axis

lenght of latus rectum is 8 and $e = \frac{3}{5}$

$$\frac{2b^2}{a} = 8 \Rightarrow \frac{b^2}{a} = 4 \Rightarrow b^2 = 4a$$
$$b^2 = a^2(1-e^2) \Rightarrow 4a = a^2\left(1 - \frac{9}{25}\right) \Rightarrow 4\cancel{a} = \cancel{a^2}^{\,a}\left(\frac{25-9}{25}\right)$$
$$4 = a\left(\frac{16}{25}\right) \Rightarrow \cancel{4} \times \frac{25}{\cancel{16}_{4}} = a \Rightarrow a = \frac{25}{4} \Rightarrow a^2 = \frac{625}{16}$$
$$b^2 = 4a$$
$$\text{where } a = \frac{25}{4}$$
$$b^2 = 4 \times \frac{25}{4} \Rightarrow b^2 = 25$$

$\therefore$ The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

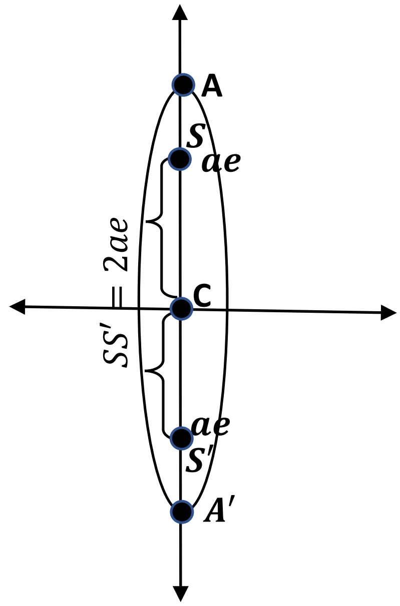
$$\frac{x^2}{\frac{625}{16}} + \frac{y^2}{25} = 1 \Rightarrow \frac{16x^2}{625} + \frac{y^2}{25} = 1$$

2(iv). Find the equation of the ellipse if the lenght of latus rectum is 4 distance between foci $4\sqrt{2}$ and major axis as $y-$axis

The major axis is $y-$axis

lenght of latus rectum is 4

$$\frac{2b^2}{a} = 4 \Rightarrow \frac{b^2}{a} = 2 \Rightarrow b^2 = 2a$$



$$Distance\ between\ foci = 4\sqrt{2} \Longrightarrow \cancel{2}ae = \overset{2}{\cancel{4}}\sqrt{2}$$

$$\boxed{ae = 2\sqrt{2}}$$

$$b^2 = a^2(1-e^2) \Longrightarrow b^2 = a^2 - a^2e^2$$

$$b^2 = a^2 - (ae)^2 \Longrightarrow 2a = a^2 - \left(2\sqrt{2}\right)^2$$

$$2a = a^2 - (4\times 2) \quad \Longrightarrow 2a = a^2 - 8$$

$$a^2 - 2a - 8 = 0 \Longrightarrow (a-4)(a+2) = 0$$

$$a - 4 = 0, a + 2 = 0$$

$$\therefore a = 4 \Longrightarrow \boxed{a^2 = 16}$$

$$b^2 = 4a\ , where\ a = 4$$

$$b^2 = 2(4) \Longrightarrow \quad b^2 = 8$$

$$\therefore The\ equation\ of\ the\ ellipse\ is\ \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\boxed{\frac{x^2}{8} + \frac{y^2}{16} = 1}$$

3 (i) Find the equation of the hyperbola if the foci are $(\pm 2, 0)$ and $e = \frac{3}{2}$

From the given data the major axis is along $x - axis$.

$$c = 2\ and\ e = \frac{1}{2}$$

$$ae = 2 \Longrightarrow a \times \frac{1}{2} = 2 \Longrightarrow a = 2\times 2$$

$$a = 4 \Longrightarrow a^2 = 16$$

$$c = 2 \Longrightarrow c^2 = 4$$

$$b^2 = a^2 + c^2 \Longrightarrow b^2 = 16 + 4$$

$$b^2 = 20$$

$$\therefore The\ equation\ of\ the\ ellipse\ is\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$

$Ex: 3(ii)$ Find the equation of the hperbola whose centre $(2, 1)$, one of the foci is $(8, 1)$, and corresponding directrix is $x = 4$.

$$centre\ (2, 1) \Longrightarrow h = 2, k = 1$$

$$foci\ (8, 1) \Longrightarrow \quad c + h = 8, k = 1$$

$$c + h = 8$$

$$c + h = 8 \Longrightarrow c + 2 = 8$$

$$c = 8 - 2 \Longrightarrow c = 6 \Longrightarrow ae = 6 \ldots (1)$$

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$$directrix: \ x = 4 \implies x = \frac{a}{e} + h$$

$$\frac{a}{e} + h = 4 \implies \frac{a}{e} + 2 = 4$$

$$\frac{a}{e} = 4 - 2 \implies \frac{a}{e} = 2 \ \ldots(2)$$

solve (1) *and* (2)

$$ae \times \frac{a}{e} = 6 \times 2 \implies a^2 = 12$$

$$b^2 = c^2 - a^2 \quad \boxed{where \ c = 6 \ and \ a^2 = 12}$$

$$b^2 = 6^2 - 12 \implies b^2 = 36 - 12 \implies b^2 = 24$$

From the given data the transverse axis is along $x - axis$.

$$\therefore The \ equation \ of \ the \ ellipse \ is \ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\boxed{\frac{x^2}{12} - \frac{y^2}{24} = 1}$$

***Ex*: 3(*ii*) *Find the equation of the hperbola passing through* (5, −2) *and lenght of transverse axis along* $x - axis$ *and of lenght* 8 *units* .**

$$lenght \ of \ transverse \ axis \ along \ x - axis = 8$$

$$2a = 8 \implies a = 4$$

$$a^2 = 16$$

$$\therefore The \ equation \ of \ the \ ellipse \ is \ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{16} - \frac{y^2}{b^2} = 1$$

it passes through (5, −2)

(x, y)

$$\frac{5^2}{16} - \frac{(-2)^2}{b^2} = 1 \implies \frac{25}{16} - \frac{4}{b^2} = 1$$

$$-\frac{4}{b^2} = 1 - \frac{25}{16} \implies -\frac{4}{b^2} = \frac{16 - 25}{16}$$

$$-\frac{4}{b^2} = \frac{16 - 25}{16} \implies -\frac{4}{b^2} = \frac{-9}{16}$$

$$\frac{4}{b^2} = \frac{9}{16} \implies 4 \times 16 = 9 \times b^2 \implies 64 = 9b^2$$

$$b^2 = \frac{64}{9}$$

$$\therefore Required \ equation \ of \ the \ ellipse \ is \ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

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$$\frac{x^2}{16}-\frac{y^2}{\frac{64}{9}}=1 \Rightarrow \frac{x^2}{16}-\frac{9y^2}{64}=1$$

4. Find the vertex, focus, equation of directrix and length of the latus rectum of the following:

(i) $y^2 = 16x$ **(ii)** $x^2 = 24y$ **(iii)** $y^2 = -8x$ **(iv)** $x^2 - 2x + 8y + 17 = 0$
(v) $y^2 - 4y - 8x + 12 = 0$

(i) $y^2 = 16x$

Compare with $y^2 = 4ax$ (*open rightward*)

$4a = 16 \Rightarrow \boxed{a = 4}$

Vertex A is $(0,0)$

focus $(a, 0) = (4, 0)$

Equation of directrix: $x = -a \Rightarrow x = -4$

Length of Latus rectum : $4a = 4(4) = 16$

(ii) $x^2 = 24y$

Compare with $x^2 = 4ay$ (*open upward*)

$4a = 24 \Rightarrow \boxed{a = 6}$

Vertex A is $(0,0)$

focus $(0, a) = (0, 6)$

Equation of directrix $y = -a \Rightarrow \boxed{y = -6}$

Length of Latus rectum : $4a = 4(6) = 24$

(iii) $y^2 = -8x$

Compare with $y^2 = -4ax$ (*open left ward*)

$4a = 8 \Rightarrow a = 2$

Vertex A is $(0,0)$

focus $(-a, 0) = (-2, 0)$

Equation of directrix $x = a \Rightarrow x = 2$

Length of Latus rectum : $4a = 4(2) = 8$

(iv) $x^2 - 2x + 8y + 17 = 0$

$x^2 - 2x + 8y + 17 = 0$ $\quad \boxed{\frac{2}{2} = 1}$

$x^2 - 2x = -8y - 17$

$\underbrace{x^2 - 2x + 1^2} - 1^2 = -8y - 17$

$(x-1)^2 - 1 = -8y - 17 \Rightarrow (x-1)^2 = -8y - 17 + 1$

$(x-1)^2 = -8y - 16$

$(x-1)^2 = -8(y+2)$

Compare with $(x-h)^2 = -4a(y-k)$ (open Downward)

$h = 1, k = -2, 4a = 8$

$\boxed{a = 2}$

$Vertex\ A\ (h,k)\ is\ (1,-2)$

$focus\ (0+h, -a+k) = (0+1, -2-2) = (1,-4)$

Equation of directrix $y = a + k \Rightarrow y = 2 - 2 \Rightarrow y = 0$

Length of Latus rectum : $4a = 4(2) = 8$

$(v)\ \boldsymbol{y^2 - 4y - 8x + 12 = 0}$

$y^2 - 4y - 8x + 12 = 0$

$y^2 - 4y = 8x - 12$

$\underbrace{y^2 - 4y + 2^2} - 2^2 = 8x - 12$

$\frac{4}{2} = 2$

$(y-2)^2 - 4 = 8x - 12$

$(y-2)^2 = 8x - 12 + 4 \Rightarrow (y-2)^2 = 8x - 8$

$(y-2)^2 = 8(x-1)$

Compare with $(y-k)^2 = 4a(x-h)$ *(open rightward)*

$h = 1, k = 2, 4a = 8$

$a = 2$

$Vertex\ A\ (h,k)\ is\ (1,2)$

$focus\ (a+h, 0+k) = (2+1, 0+2) = (3,2)$

Equation of directrix $x = -a + h \Rightarrow x = -2 + 1 \Rightarrow \boxed{x = -1}$

Length of Latus rectum : $4a = 4(2) = 8$

5. *Identify the type of conic and find centre, foci, vertices, and directrices of each of the following*:

$(i)\ \frac{x^2}{25} + \frac{y^2}{9} = 1 \quad (ii)\ \frac{x^2}{3} + \frac{y^2}{10} = 1 \quad (iii)\ \frac{x^2}{25} - \frac{y^2}{144} = 1 \quad (iv)\ \frac{y^2}{16} - \frac{y^2}{9} = 1$

$\frac{x^2}{25} + \frac{y^2}{9} = 1$ (*Given conic is ellipse*)

Compare with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *(major axis is along X– axis)*

$a^2 = 25, b^2 = 9 \Rightarrow a = 5, b = 3$

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$c^2 = a^2 - b^2 \Rightarrow c = \sqrt{a^2 - b^2}$

$c = \sqrt{5^2 - 3^2} \Rightarrow c = \sqrt{25 - 9} \Rightarrow c = \sqrt{16}$

$\boxed{c = 4}$

Centre: $(0, 0)$

Foci: $(\pm c, 0) \Rightarrow (\pm 4, 0)$

Vertices : $(\pm a, 0) \Rightarrow (\pm 5, 0)$

To Find eccentricity (e)

$e = \frac{c}{a} \Rightarrow e = \frac{4}{5}$

$\frac{a}{e} = \frac{5}{\frac{4}{5}} = 5 \times \frac{5}{4} \Rightarrow \frac{a}{e} = \frac{25}{4}$

Equation of directrices : $x = \pm \frac{a}{e} \Rightarrow x = \pm \frac{25}{4}$

$(\boldsymbol{ii})\ \frac{\boldsymbol{x^2}}{\boldsymbol{3}} + \frac{\boldsymbol{y^2}}{\boldsymbol{10}} = \boldsymbol{1}$ (*Given conic is ellipse*)

Compare with $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ *(major axis is along y– axis)*

$b^2 = 3, \quad a^2 = 10$

$b = \sqrt{3}, \quad a = \sqrt{10}$

$c^2 = a^2 - b^2 \Rightarrow c = \sqrt{a^2 - b^2}$

$c = \sqrt{10 - 3} \Rightarrow \boxed{c = \sqrt{7}}$

To Find eccentricity (e)

$e = \frac{c}{a} \Rightarrow e = \frac{\sqrt{7}}{\sqrt{10}}$

Centre: $(0, 0)$

Foci: $(0, \pm c) \Rightarrow (0, \pm \sqrt{7})$

Vertices : $(0, \pm a) \Rightarrow (0, \pm \sqrt{10})$

To Find $\frac{a}{e}$

$\frac{a}{e} = \frac{\sqrt{10}}{\frac{\sqrt{7}}{\sqrt{10}}} = \sqrt{10} \times \frac{\sqrt{10}}{\sqrt{7}} = \frac{10}{\sqrt{7}}$

where $a = \sqrt{10}$ *and* $e = \frac{\sqrt{7}}{\sqrt{10}}$

$\boxed{\frac{a}{e} = \frac{10}{\sqrt{7}}}$

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Equation of directrices : $y = \pm\frac{a}{e} \Rightarrow y = \pm\frac{10}{\sqrt{7}}$

$(iii)\ \frac{x^2}{25} - \frac{y^2}{144} = 1$

Compare with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *(transverse axis is along X– axis)*

$a^2 = 25, b^2 = 144 \Rightarrow a = 5 , b = 12$

$c^2 = a^2 + b^2 \Rightarrow c = \sqrt{a^2 + b^2}$

$c = \sqrt{a^2 + b^2} \Rightarrow c = \sqrt{25 + 144} \Rightarrow c = \sqrt{169}$

$\boxed{c = 13}$

Centre: $(0, 0)$

Foci: $(\pm c, 0) \Rightarrow (\pm 13, 0)$

Vertices : $(\pm a, 0) \Rightarrow (\pm 5, 0)$

$\boxed{To\ Find\ eccentricity\ (e)}$

$e = \frac{c}{a} \Rightarrow \boxed{e = \frac{13}{5}}$

$To\ Find\ \frac{a}{e}$

$\frac{a}{e} = \frac{5}{\frac{13}{5}} = 5 \times \frac{5}{13} = \frac{25}{13}$

$where\ a = 5\ and\ e = \frac{13}{5}$

$\boxed{\frac{a}{e} = \frac{25}{13}}$

Equation of directrices : $x = \pm\frac{a}{e} \Rightarrow x = \pm\frac{25}{13}$

$(iv)\ \frac{y^2}{16} - \frac{y^2}{9} = 1$

Compare with $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ *(transverse axis is along X– axis)*

$a^2 = 16, b^2 = 9 \Rightarrow a = 4 , b = 3$

$c^2 = a^2 + b^2 \Rightarrow c = \sqrt{16 + 9}$

$c = \sqrt{25} \Rightarrow \boxed{c = 5}$

Centre: $(0, 0)$

Foci: $(0, \pm c) \Rightarrow (0, \pm 5)$

Vertices : $(0, \pm a) \Rightarrow (0, \pm 4)$

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To Find eccentricity (e)

$$e = \frac{c}{a} \Rightarrow \boxed{e = \frac{5}{4}}$$

To Find $\frac{a}{e}$

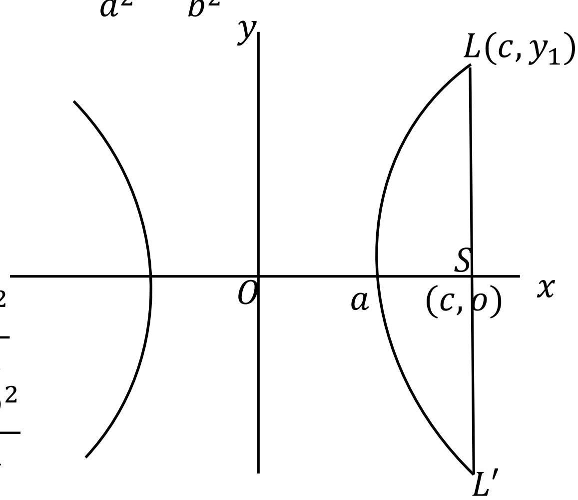
$$\frac{a}{e} = \frac{4}{\frac{5}{4}} = 4 \times \frac{4}{5} = \frac{16}{5} \Rightarrow \boxed{\frac{a}{e} = \frac{16}{5}}$$

Equation of directrices : $y = \pm\frac{a}{e} \Rightarrow y = \pm\frac{16}{5}$

6. Prove that the length of the latus rectum of the hyperbola

Equation of hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The point (c, y_1) lies on the hyperbola $\frac{c^2}{a^2} - \frac{{y_1}^2}{b^2} = 1$



$$\frac{c^2}{a^2} - 1 = \frac{{y_1}^2}{b^2} \Rightarrow \frac{{y_1}^2}{b^2} = \frac{c^2 - a^2}{a^2}$$

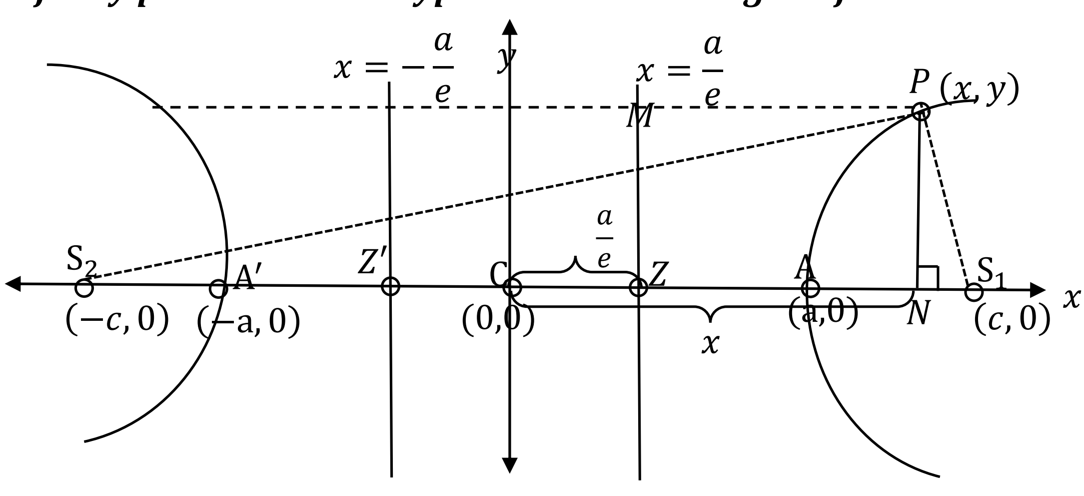
where $c^2 = a^2 + b^2$

$$\frac{{y_1}^2}{b^2} = \frac{a^2 + b^2 - a^2}{a^2} \Rightarrow \frac{{y_1}^2}{b^2} = \frac{b^2}{a^2}$$

$${y_1}^2 = \frac{b^4}{a^2} \Rightarrow y_1 = \sqrt{\frac{b^4}{a^2}} \Rightarrow y_1 = \frac{b^2}{a}$$

Length of latus rectum $= 2y_1 = \frac{2b^2}{a}$

7. Show that the absolute value of difference of the focal distances of any point P on the hyperbola is the length of its transverse axis.



Definition of a Conic : $\frac{SP}{PM} = e$

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$SP = ePM \implies SP = e(CN - CZ)$

$SP = e\left(x - \frac{a}{e}\right) \implies SP = xe - \frac{a}{e} \times e$

$\boxed{SP = xe - a}$

$SP' = ePM' \implies SP' = e(CN + CZ')$

$SP' = e\left(x + \frac{a}{e}\right) \implies SP' = xe + \frac{a}{e} \times e$

$SP' = xe + a$

$SP' - SP = xe + a - xe + a = 2a$

8. Identify the type of conic and find centre, foci, vertices, and directrices of each of the following :

$(i)\ \frac{(x-3)^2}{225} + \frac{(y-4)^2}{289} = 1 \quad (ii)\ \frac{(x+1)^2}{100} + \frac{(y-2)^2}{64} = 1$

$(i)\ \frac{(x-3)^2}{225} + \frac{(y-4)^2}{289} = 1$

Compare with $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ (major axis is along y– axis)

$h = 3, k = 4$

$b^2 = 225, a^2 = 289 \implies b = 15, a = 17$

$c^2 = a^2 - b^2 \implies c = \sqrt{a^2 - b^2}$

$c = \sqrt{289 - 225} \implies c = \sqrt{64}$

$\boxed{c = 8}$

To Find eccentricity (e)

$e = \frac{c}{a} \implies e = \frac{8}{17}$

centre $(h, k) \implies (3, 4)$

vertices $(0 + h, \pm a + k) = (3, \pm 17 + 4) = (3, 17 + 4), (3, -17 + 4)$

$= (3, 21), (3, -13)$

foci: $(0 + h, \pm c + k) = (3, \pm 8 + 4) = (3, 8 + 4), (3, -8 + 4)$

$= (3, 12), (3, -4)$

To Find $\frac{a}{e}$

$\frac{a}{e} = \frac{17}{\frac{8}{17}} = 17 \times \frac{17}{8} = \frac{289}{8}$

$\boxed{\frac{a}{e} = \frac{289}{8}}$

Equation of directrices : $y = \pm\frac{a}{e} + k$

$$y = \pm\frac{289}{8} + 4 \Rightarrow y = \frac{289}{8} + 4, -\frac{289}{8} + 4$$

$$y = \frac{289 + 32}{17}, -\frac{-289 + 32}{17} \Rightarrow y = \frac{321}{17}, -\frac{257}{17}$$

$$(ii)\ \frac{(x+1)^2}{100} + \frac{(y-2)^2}{64} = 1$$

Compare with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ *(major axis is along x− axis)*

$$h = -1, k = 2$$

$$a^2 = 100, b^2 = 64 \Rightarrow a = 10, b = 8$$

$$c^2 = a^2 - b^2 \Rightarrow c = \sqrt{a^2 - b^2}$$

$$c = \sqrt{100 - 64} \Rightarrow c = \sqrt{36}$$

$$\boxed{c = 6}$$

To Find eccentricity (e)

$$e = \frac{c}{a} \Rightarrow e = \frac{6}{10} \Rightarrow \boxed{e = \frac{3}{5}}$$

$$centre\ (h, k) \Rightarrow (-1, 2)$$

$$vertices\ (\pm a + h, 0 + k) = (\pm 10 - 1, 2) = (10 - 1, 2), (-10 - 1, 2)$$

$$= (9, 2), (-11, 2)$$

$$foci: (\pm c + h, 0 + k) = (\pm 6 - 1, 2) = (6 - 1, 2), (-6 - 1, 2)$$

$$= (5, 2), (-7, 2)$$

To Find $\boldsymbol{\frac{a}{e}}$

$$\frac{a}{e} = \frac{10}{\frac{3}{5}} = 10 \times \frac{5}{3} = \frac{50}{3}$$

$$where\ a = 10\ and\ e = \frac{3}{5}$$

$$\boxed{\frac{a}{e} = \frac{50}{3}}$$

Equation of directrices: $x = \pm\frac{a}{e} + h$

$$x = \pm\frac{50}{3} - 1 \Rightarrow x = \frac{50}{3} - 1, x = -\frac{50}{3} - 1$$

$$x = \frac{50 - 3}{3}, \frac{-50 - 3}{3} \Rightarrow x = \frac{47}{3}, \frac{-53}{3}$$

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$$(iii)\ \frac{(x+3)^2}{225}-\frac{(y-4)^2}{64}=1$$

Compare with $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ (transverse axis is along X− axis)

$h=-3, k=4$

$a^2=225, b^2=64 \implies a=15,\ b=8$

$c^2=a^2+b^2 \implies c=\sqrt{a^2+b^2}$

$c=\sqrt{225+64} \implies c=\sqrt{289}$

$\boxed{c=17}$

To Find eccentricity (e)

$e=\frac{c}{a} \implies e=\frac{17}{15}$

centre $(h,k) \implies (-3,4)$

vertices $(\pm a+h, 0+k)=(\pm 15-3,4)=(15-3,2),(-15-3,2)$

$=(12,2),(-18,2)$

foci: $(\pm c+h,0+k)=(\pm 17-3,4)=(17-3,4),(-17-3,4)$

$=(14,4),(-20,4)$

To Find $\frac{a}{e}$

$$\frac{a}{e}=\frac{15}{\frac{17}{15}}=15\times\frac{15}{17}=\frac{225}{17}$$

where $a=15$ and $e=\frac{17}{15}$

$$\boxed{\frac{a}{e}=\frac{225}{17}}$$

Equation of directrices: $x=\pm\frac{a}{e}+h$

$x=\pm\frac{225}{17}-3 \implies x=\frac{225}{17}-3,\ x=-\frac{225}{17}-3$

$x=\frac{225-51}{17},\frac{-225-51}{17} \implies x=\frac{174}{17},\frac{-276}{17}$

$$(iv)\ \frac{(y-2)^2}{25}-\frac{(x+1)^2}{16}=1$$

Compare with $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ (transverse axis is along X− axis)

$a^2=25, b^2=16 \implies a=5, b=4$

$c^2=a^2+b^2 \implies c=\sqrt{25+16}$

$\boxed{c=\sqrt{41}}$

$h=-1, k=2$

Centre: $(h,k) = (-1,2)$

Foci: $(0+h, \pm c+k) \Rightarrow (-1, \pm\sqrt{41}+2)$

$\left(-1, \sqrt{41}+2\right), \left(-1, -\sqrt{41}+2\right)$

Vertices : $(0+h, \pm a+k) \Rightarrow (-1, \pm 5 + 2)$

$(-1, 5+2),\ (-1, -5+2) = (-1, 7), (-1, -3)$

To Find eccentricity (e)

$e = \frac{c}{a} \Rightarrow \boxed{e = \frac{\sqrt{41}}{5}}$

To Find $\frac{a}{e}$

$\frac{a}{e} = \frac{5}{\frac{\sqrt{41}}{5}} = 5 \times \frac{5}{\sqrt{41}} = \frac{25}{\sqrt{41}}$

$\frac{a}{e} = \frac{25}{\sqrt{41}}$

Equation of directrices : $y = \pm\frac{a}{e} + k$

$y = \pm\frac{25}{\sqrt{41}} + 2 \Rightarrow y = \frac{25}{\sqrt{41}} + 2, -\frac{25}{\sqrt{41}} - 2$

(v) $\mathbf{18x^2 + 12y^2 - 144x + 48y + 120 = 0}$

$18x^2 - 144x + 12y^2 + 48y + 120 = 0$

$18(x^2 - 8x) + 12(y^2 + 4y) + 120 = 0$

$18(\underbrace{x^2 - 8x + 4^2} - 4^2) + 12\,(\underbrace{y^2 + 4y + 2^2} - 2^2) + 120 = 0$

$18[(x-4)^2 - 16] + 12[(y+2)^2 - 4] + 120 = 0$

$18(x-4)^2 - 288 + 12(y+2)^2 - 48 + 120 = 0$

$18(x-4)^2 + 12(y+2)^2 - 216 = 0$

$18(x-4)^2 + 12(y+2)^2 = 216$

$\div 216$

$\frac{(x-4)^2}{12} + \frac{(y+2)^2}{18} = 1$

$\frac{(x-4)^2}{12} + \frac{(y+2)^2}{18} = 1$

Compare with $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ *(major axis is along y– axis)*

$h = 4, k = -2$

$b^2 = 12, a^2 = 18 \Rightarrow b = \sqrt{12}, a = \sqrt{18} \Rightarrow b = 2\sqrt{3}, a = 3\sqrt{2}$

$c^2 = a^2 - b^2 \Longrightarrow c = \sqrt{a^2 - b^2}$

$c = \sqrt{18 - 12} \Longrightarrow \boxed{c = \sqrt{6}}$

To Find eccentricity (e)

$e = \frac{c}{a} \Longrightarrow e = \frac{\sqrt{6}}{3\sqrt{2}} \Longrightarrow e = \frac{\sqrt{3} \times \sqrt{2}}{3\sqrt{2}} \Longrightarrow e = \frac{\sqrt{3}}{3}$

$e = \frac{\sqrt{3}}{\sqrt{3} \times \sqrt{3}} \Longrightarrow e = \frac{1}{\sqrt{3}}$

$centre\ (h,k) \Longrightarrow (4,-2)$

$vertices: (0 + h, \pm a + k) = (4, \pm 3\sqrt{2} - 2) = (4, 3\sqrt{2} - 2), (4, -3\sqrt{2} - 2)$

$foci: (0 + h, \pm c + k) = (4, \pm\sqrt{6} - 2) = (4, \sqrt{6} - 2), (4, \sqrt{6} - 2)$

$To\ Find\ \frac{a}{e}$

$\frac{a}{e} = \frac{3\sqrt{2}}{\frac{1}{\sqrt{3}}} = 3\sqrt{2} \times \sqrt{3} = 3\sqrt{6}$

$\boxed{\frac{a}{e} = 3\sqrt{6}}$

Equation of directrices : $y = \pm\frac{a}{e} + k$

$y = \pm 3\sqrt{6} - 2 \Longrightarrow y = 3\sqrt{6} - 4, -3\sqrt{6} - 2$

$(vi)\ \mathbf{9x^2 - y^2 - 36x - 6y + 18 = 0}$

$9x^2 - 36x - y^2 - 6y + 18 = 0$

$9(x^2 - 4x) - (y^2 + 6y) + 19 = 0$

$9(x^2 - 4x + 2^2 - 2^2) - (y^2 + 6y + 3^2 - 3^2) + 18 = 0$

$9[(x-2)^2 - 4] - [(y+3)^2 - 9] + 18 = 0$

$9(x-2)^2 - 36 - (y+3)^2 + 9 + 18 = 0$

$9(x-2)^2 - (y+3)^2 - 36 + 27 = 0$

$9(x-2)^2 - (y+3)^2 - 9 = 0$

$9(x-2)^2 - (y+3)^2 = 9$

$\div 9$

$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$

$compare\ with\ \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \therefore [transerve\ axis\ along\ x - axis]$

$h = 2\ , k = -3$

$a^2 = 1, b^2 = 9 \Longrightarrow a = 1, b = 3$

$centre : (h, k) = (2, -3)$

$vertices : (\pm a + h , 0 + k) \Rightarrow (\pm 1 + 2 , 0 - 3)$

$= (1 + 2 , 0 - 3) = (3 , -3)$

$c^2 = a^2 + b^2 \Rightarrow c = \sqrt{a^2 + b^2}$

$c = \sqrt{1^2 + 3^2} \Rightarrow c = \sqrt{1 + 9} \Rightarrow \boxed{c = \sqrt{10}}$

$foci : (\pm c + h , 0 + k) \Rightarrow (\pm \sqrt{10} + 2 , 0 - 3)$

$= \left(\sqrt{10} + 2 - 3\right), (-\sqrt{10} + 2 , -3)$

$To\ find : (e)$

$e = \frac{c}{a} \Rightarrow e = \frac{\sqrt{10}}{1} \Rightarrow \boxed{e = \sqrt{10}}$

$To\ find : \frac{a}{e}$

$\frac{a}{e} = \frac{1}{\sqrt{10}}$ where $a = 1$ and $e = \sqrt{10}$

$Directrices : x = \pm \frac{a}{e} + h = \pm \frac{1}{\sqrt{10}} + 2 = \frac{1}{\sqrt{10}} + 2 , -\frac{1}{\sqrt{10}} + 2$

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EXERCISE : 5.4

Example 5.27 Find the equations of tangent and normal to the parabola $x^2+6x+4y+5=0$ ***at*** $(1,-3)$.

Equation of parabola is $x^2+6x+4y+5=0$

Equation of tangent to the parabola is $x^2+6x+4y+5=0$ *at* $(1,-3)$ (x_1,y_1)

$$xx_1+6\left(\frac{x+x_1}{2}\right)+4\left(\frac{y+y_1}{2}\right)+5=0$$

$$x(1)+\overset{3}{\cancel{6}}\left(\frac{x+1}{\cancel{2}}\right)+\overset{2}{\cancel{4}}\left(\frac{y-3}{\cancel{2}}\right)+5=0 \Longrightarrow x+3(x+1)+2(y-3)+5=0$$

$$x+3x+3+2y-6+5=0 \Longrightarrow 4x+2y+2=0$$

Equations of tangent $2x+y+1=0$

Equation of normal is of the form $x-2y+k=0$

It pases through $(1,-3)$

$$1-2(-3)+k=0 \Longrightarrow 1+6+k=0 \Longrightarrow 7+k=0$$

$$\boxed{k=-7}$$

Equation of normal is $x-2y-7=0$

Example 5.28: Find the equation of tangent and normal to the ellipse $x^2+4y^2=32$ ***when*** $\theta=\frac{\pi}{4}$

Equation of ellipse is $x^2+4y^2=32$

$\div 32$

$$\frac{x^2}{32}+\frac{y^2}{8}=1$$

$$a^2=32, b^2=8 \Longrightarrow a=4\sqrt{2}, b=2\sqrt{2}$$

The parametric form of the ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is* $(a\cos\theta, b\sin\theta)$

$$At, \theta=\frac{\pi}{4}$$

$$(a\cos\theta, b\sin\theta)=\left(4\sqrt{2}\cos\frac{\pi}{4}, 2\sqrt{2}\sin\frac{\pi}{4}\right)=\left(4\cancel{\sqrt{2}}\times\frac{1}{\cancel{\sqrt{2}}}\ 2\cancel{\sqrt{2}}\times\frac{1}{\cancel{\sqrt{2}}}\right)$$

$$=(4,2)$$

$\therefore$ *Equation of tangent to the ellipse* $x^2+4y^2=32$ *at* $(4,2)$ (x_1,y_1)

$$xx_1+4yy_1=32$$

$$x(4)+4y(2)=32 \Longrightarrow 4x+8y=32$$

$\therefore$ ***Equation of tangent is*** $x+2y-8=0$

Equation of normal is of the form $2x-y+k=0$

It pases through $(4,2)$

$$2(4)-2+k=0 \Longrightarrow 8-2+k=0 \Longrightarrow 6+k=0$$

$$\boxed{k=-6}$$

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1. Find the equation of the two tangents that can be drawn from the point $(5,2)$ to the ellipse $2x^2+7y^2=14$

Ellipse: $2x^2+7y^2=14 \Rightarrow \frac{2x^2}{14}+\frac{7y^2}{14}=1$ $\div 14$

$$\frac{x^2}{7}+\frac{y^2}{2}=1$$

Compare with : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow a^2=7, b^2=2$

Equation of any tangent is of the form : $y=mx\pm\sqrt{a^2m^2+b^2}$

$$y=mx\pm\sqrt{7m^2+2}$$

It passes through the point $(5,2)$

$$2=m\times 5\pm\sqrt{7m^2+2} \Rightarrow 2-5m=\pm\sqrt{7m^2+2}$$

Squaring on both sides

$$(2-5m)^2=7m^2+2 \Rightarrow 4-20m+25m^2=7m^2+2$$

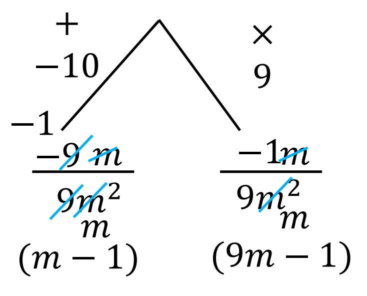
$$4-20m+25m^2-7m^2-2=0$$

$$18m^2-20m+2=0$$

$\div 2$

$$9m^2-10m+1=0 \Rightarrow (9m-1)(m-1)=0$$

$$9m-1=0,\ m-1=0 \Rightarrow m=\frac{1}{9}, m=1$$



The equation of the tangents at $(5,2)$ (x_1,y_1)

is $y-y_1=m(x-x_1)$

$$y-2=m(x-5)$$

When $m=\frac{1}{9} \Rightarrow y-2=\frac{1}{9}(x-5)$

$$9(y-2)=1(x-5) \Rightarrow 9y-18=x-5$$

$$x-5-9y+18=0 \Rightarrow x-9y+13=0$$

When $\boldsymbol{m=1}$

$$y-2=1(x-5)$$

$$y-2=x-5 \Rightarrow x-5-y+2=0$$

$$\boxed{x-y-3=0}$$

2. Find the equation of tangents to the hyperbola $\frac{x^2}{16}-\frac{y^2}{64}=1$ which are parallel to $10x-3y+9=0$

Hyperbola: $\frac{x^2}{16}-\frac{y^2}{64}=1$

Compare with : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$a^2=16 \Rightarrow a=4$ *and* $b^2=64 \Rightarrow b=8$

Equation of any tangent is of the form $: \boldsymbol{y = mx \pm \sqrt{a^2m^2 - b^2}}$

$$y = mx \pm \sqrt{16m^2 - 64} \rightarrow ①$$

Slope of the line $ax + by + c = 0$ *is* $m = -\frac{coefficient\ of\ x}{coefficient\ of\ y}$

Slope of the line $10x - 3y + 9 = 0$ *is* $m = \frac{-10}{-3} = \frac{10}{3}$

sub $m = \frac{10}{3}$ *in* (1) $y = mx \pm \sqrt{16m^2 - 64}$

$$y = \frac{10}{3} \times x \pm \sqrt{16\left(\frac{10}{3}\right)^2 - 64} \Rightarrow y = \frac{10}{3}x \pm \sqrt{16 \times \frac{100}{9} - 64}$$

$$y = \frac{10}{3}x \pm \sqrt{\frac{1600}{9} - 64} \Rightarrow y = \frac{10}{3}x \pm \sqrt{\frac{1600 - 64 \times 9}{9}}$$

$$y = \frac{10}{3}x \pm \sqrt{\frac{1600 - 576}{9}} \Rightarrow y = \frac{10}{3}x \pm \sqrt{\frac{1024}{9}}$$

$$y = \frac{10}{3}x \pm \frac{32}{3} \Rightarrow y = \frac{10x \pm 32}{3} \Rightarrow 3y = 10x \pm 32$$

$$3y = 10x + 32,\ 3y = 10x - 32$$

Required tangents are $\boldsymbol{10x - 3y + 32 = 0, 10x - 3y - 32 = 0}$

3. *Show that the line* $\boldsymbol{x - y + 4 = 0}$ ***is a tangent to the ellipse*** $\boldsymbol{x^2 + 3y^2 = 12}$***. Also find the co − ordinates of the point of the contact.***

Ellipse: $x^2 + 3y^2 = 12 \Rightarrow \frac{x^2}{12} + \frac{3y^2}{12_4} = 1$
$\div 12$

$$\frac{x^2}{12} + \frac{y^2}{4} = 1$$

Compare with $: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow a^2 = 12, b^2 = 4$

Condition for $\boldsymbol{y = mx + c}$ **to be a tangent to** $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ ***is*** $\boldsymbol{c = \pm\sqrt{a^2m^2 + b^2}}$

Line: $x - y + 4 = 0 \Rightarrow y = x + 4$

Compare with $y = mx + c$

$$m = 1, c = 4$$

$$\boxed{c = 4 \ldots (1)}$$

Sub $a^2 = 12, b^2 = 4$ and $m = 1$ *in* $c = \pm\sqrt{a^2m^2 + b^2}$

$$c = \pm\sqrt{12(1)^2 + 4} \Rightarrow c = \pm\sqrt{12 + 4} \Rightarrow c = \pm\sqrt{16}$$

$\boxed{c = 4}$ ... (2)

From (1) and (2) the line $x - y + 4 = 0$ *is a tangent to the ellipse*

$$\boxed{\text{The point of contact is } \left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)}$$

$$\frac{-a^2m}{c} = \frac{-12 \times 1}{4} = -3 \text{ and } \frac{b^2}{c} = \frac{4}{4} = 1$$

$$\boxed{\text{The point of contact is } (-3, 1)}$$

4. Find the equation of the tangent to the parabola $y^2 = 16x$ perpendicular to $2x + 2y + 3 = 0$

Parabola: $y^2 = 16x$

Compare with $y^2 = 4ax$

$4a = 16 \qquad a = 4$

Line: $2x + 2y + 3 = 0 \qquad 2y = -2x - 3$
$\div 2$

$$y = -x + \frac{3}{2}$$

Comparing : $y = mx + c \implies m = -1$, *and* $c = \frac{3}{2}$

slope of perpendicular $= \frac{-1}{m} = \frac{-1}{-1}$ $\therefore$ *slope of perpendicular* $= 1$

$y = mx + c$ *is the tangent to the parabola if* $c = \frac{a}{m}$

$$c = \frac{4}{1} \implies c = 4$$

sub $m = 1, c = 4$ *in* $y = mx + c$

$$y = x + 4 \implies x - y + 4 = 0$$

5. Find the equation of the tangent at $t = 2$ to the parabola $y^2 = 8x$ (*Hint*: use parametric form)

Parabola : $y^2 = 8x$

Compare with $y^2 = 4ax$

$4a = 8 \implies \boxed{a = 2}$

The parametric form of the parabola: $y^2 = 4ax$ *is* $(at^2, 2at)$

$(at^2, 2at)$ *where* $t = 2$ *and* $a = 2$

$(2(2)^2, 2 \times 2 \times 2) \implies (8, 8)$

The equation of the tangent at (x_1, y_1) to the parabola $y^2 = 4ax$ is

$\boxed{yy_1 = 4a\left(\frac{x + x_1}{2}\right)}$ *Here* $(x_1, y_1) = (8, 8)$

$$y(8) = 4(2)\left(\frac{x+8}{2}\right) \implies 8y = 4(x+8) \implies 2y = x + 8$$
$\div 4$

Equation of tangent is $x - 2y + 8 = 0$

6. Find the equations of the tangent and normal to hyperbola $12x^2 - 9y^2 = 108$ at $\theta = \frac{\pi}{3}$.

Hyperbola: $12x^2 - 9y^2 = 108$

$\div 108$

$$\frac{12x^2}{108} - \frac{9y^2}{108} = 1 \Rightarrow \frac{x^2}{9} - \frac{y^2}{12} = 1$$

Compare with $: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow a^2 = 9, b^2 = 12$

$$a = \sqrt{9}, b = \sqrt{12} \Rightarrow a = 3, b = 2\sqrt{3}$$

The parametric equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *is* $(asec\theta, btan\theta)$

When $\theta = \frac{\pi}{3}$

$$x = asec\theta = 3sec\left(\frac{\pi}{3}\right) = 3 \times \frac{2}{1} = 6$$

$$y = btan\theta = 2\sqrt{3}tan\left(\frac{\pi}{3}\right) = 2\sqrt{3} \times \frac{\sqrt{3}}{1} = 6$$

$\therefore$ *The point is* $(6, 6)$

The equation of the tangent at (x_1, y_1) *to the given ellipse is*
$12xx_1 - 9yy_1 = 108$

Here $(x_1, y_1) = (6, 6)$

$$12x(6) - 9y(6) = 108 \Rightarrow 72x - 54y - 108 = 0$$

$\div 18$

The equation of tangent is $4x - 3y - 6 = 0$

Equation of normal is of the form $3x + 4y + k = 0$

It pases through $(6,6)$

$$3(6) + 4(6) + k = 0 \Rightarrow 18 + 24 + k = 0$$

$$42 + k = 0 \Rightarrow \boxed{k = -42}$$

The equation of normal is $3x + 4y - 42 = 0$

7. Prove that the point of intersection of the tangents at 't_1' and 't_2' on the parabola $y^2 = 4ax$ is $[at_1t_2, a(t_1 + t_2)]$.

Parametric equation of tangent at t_1 *is* $yt_1 = x + at_1^2 \ldots (1)$

Parametric equation of tangent at t_2 *is* $yt_2 = x + at_2^2 \ldots (2)$

Solve (1) and (2)

$(1) \Rightarrow \quad yt_1 = x + at_1^2$

$\quad (-) \quad (-) \quad (-)$

$(2) \Rightarrow \quad yt_2 = x + at_2^2$

$$yt_1 - yt_2 = at_1^2 - at_2^2 \Rightarrow y(t_1 - t_2) = a(t_1^2 - t_2^2)$$

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$$y(t_1 - t_2) = a(t_1 - t_2)(t_1 + t_2) \Rightarrow \boxed{y = a(t_1 + t_2)}$$

$$Sub\ y = a(t_1 + t_2)\ (1)\ yt_1 = x + at_1^2$$

$$a(t_1 + t_2)t_1 = x + at_1^2 \Rightarrow (at_1 + at_2)t_1 = x + at_1^2$$

$$at_1^2 + at_1t_2 = x + at_1^2 \Rightarrow \boxed{x = at_1t_2}$$

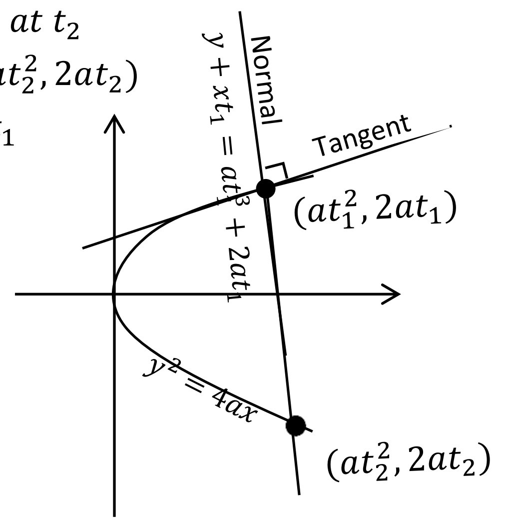
The point of intersection of tangents at t_1 *and* t_2 *on the parabola is* $[at_1t_2, a(t_1 + t_2)]$

8. If the normal at the point 't_1' on the parabola $y^2 = 4ax$ meets the parabola again at the point, then prove that $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$

Equation of normal at t_1 *to the parabola is* $y + xt_1 = at_1^3 + 2at_1$

Above normal again meets the parabola at t_2

So the meeting point on the parabola is $(at_2^2, 2at_2)$



$$Sub(at_2^2, 2at_2)\ in\ (1)\ y + xt_1 = at_1^3 + 2at_1$$

$$2at_2 + (at_2^2)t_1 = at_1^3 + 2at_1$$

$$2at_2 + at_1t_2^2 = at_1^3 + 2at_1$$

$$2at_2 - 2at_1 = at_1^3 - at_1t_2^2$$

$$2a(t_2 - t_1) = -at_1(t_2^2 - t_1^2)$$

$$2a(t_2 - t_1) = -at_1(t_2^2 - t_1^2)$$

$$2a(t_2 - t_1) = -at_1(t_2 - t_1)(t_2 + t_1)$$

$$2 = -t_1(t_2 + t_1)$$

$$-\frac{2}{t_1} = t_2 + t_1 \Rightarrow t_2 = -t_1 - \frac{2}{t_1} \Rightarrow t_2 = -\left(t_1 + \frac{2}{t_1}\right)$$

Hence Proved

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Exercise : 5.5

Example 5.30 A semielliptical archway over a one~way road has a height of 3m and a width of 12m. The truck has a width of 3m and a height of 2.7m. Will the truck clear the opening of the archway?

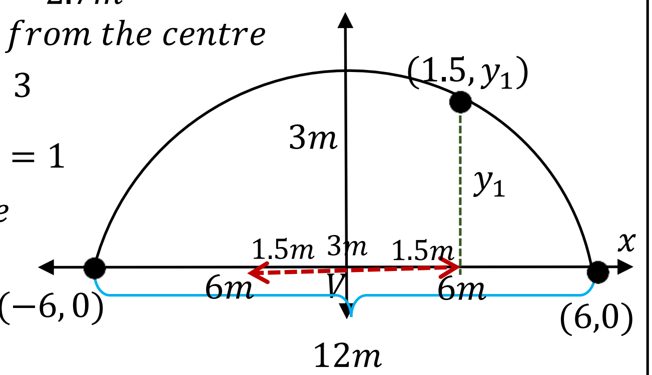
Truck's width $= 3m$, *Truck's height* $= 2.7m$

To find height of the archway $1.5m$ *from the centre*

From the diagram $a = 6$ *and* $b = 3$

The equation of ellipse as $\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1$

The point $(1.5, y_1)$ *lies on the ellipse*



$$\frac{\left(\frac{3}{2}\right)^2}{36} + \frac{{y_1}^2}{9} = 1 \Rightarrow \frac{\frac{9}{4}}{36} + \frac{{y_1}^2}{9} = 1$$

$$\frac{9}{4} \times \frac{1}{36} + \frac{{y_1}^2}{9} = 1 \Rightarrow \frac{9}{4} \times \frac{1}{36_{4}} + \frac{{y_1}^2}{9} = 1$$

$$\frac{1}{16} + \frac{{y_1}^2}{9} = 1 \Rightarrow \frac{{y_1}^2}{9} = 1 - \frac{1}{16}$$

$$\frac{{y_1}^2}{9} = \frac{16-1}{16} \Rightarrow \frac{{y_1}^2}{9} = \frac{15}{16} \Rightarrow {y_1}^2 = \frac{15}{16} \times 9$$

$$y_1 = \sqrt{\frac{15}{16} \times 9} \Rightarrow y_1 = \frac{3}{4}\sqrt{15}$$

$$y_1 = \frac{3}{4} \times 3.87 \Rightarrow y_1 = \frac{11.61}{4}$$

$$y_1 = 2.90$$

	3·87
3	15
	9
68	600
	544
767	5600
	5348
	252

The height of arch way $1.5m$ *from the centre is approximately* $2.90m$.

Example: 5.31 The maximum and minimum distances of the Earth from the Sunrespectivelyare 152×10^6 ***km and*** $94.5 10^6$***km. The Sun is at one focus of the elliptical orbit . Find the distance from the Sun to the other focus.***

$AS = 94.5 \times 10^6\ km, \quad A'S = 152 \times 10^6\ km$

minimum distance between the earth and sun

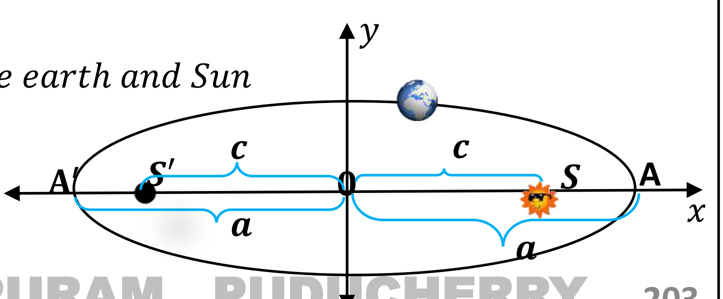
$AS = a - c$

$a - c = 94.5 \times 10^6 \ldots (1)$

maximum distance between the earth and Sun

$A'S = a + c$

$a + c = 152 \times 10^6 \ldots (2)$



solve (1) *and* (2)

$$\begin{array}{l} a - c = 94.5 \times 10^6 \\ (-)\ \ (-)\ \ (-) \\ a + c = 152 \times 10^6 \\ \hline \end{array}$$

$$-2c = -57.5 \times 10^6 \Longrightarrow 2c = 57.5 \times 10^6$$

$$2c = 57.5 \times 10^6 \Longrightarrow 2c = 575 \times 10^5$$

Distance of the Sun from the other focus is $SS' = 575 \times 10^5\ km$.

Example 5.32 A concrete bridge is designed as parabolic arch. The road over bridge is 40m long and the maximum height of the arch is 15m. Write the equation of the parabolic arch?.

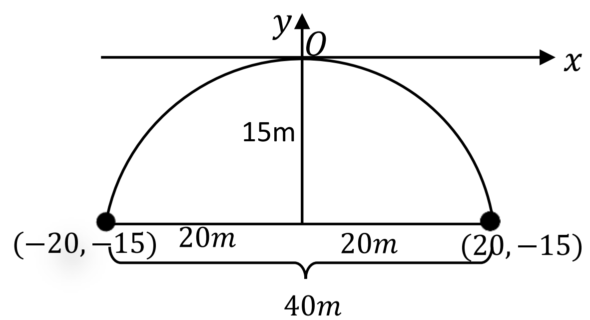
A bridge of parabolic arc has open downward $x^2 = -4ay$ ... (1)

The point $(20, -15)$ *lie on the parabola*

$$x^2 = -4ay \Longrightarrow 20^2 = -4a(-15)$$

$$400 = 60a \Longrightarrow a = \frac{400}{60} = \frac{20}{3}$$

$$a = \frac{20}{3}$$



Sub $a = \dfrac{20}{3}$ in $x^2 = -4ay$

$$x^2 = -4\left(\frac{20}{3}\right)y \Longrightarrow x^2 = -\frac{80}{3}y$$

$\therefore$ ***Equation is*** $\mathbf{3x^2 = -80y}$

Example 5.33 The parabolic communication antenna has a focus at 2m distance from the vertex of the antenna. Find the width of the antenna 3m from the vertex.

Equation of parabola is $y^2 = 4ax$

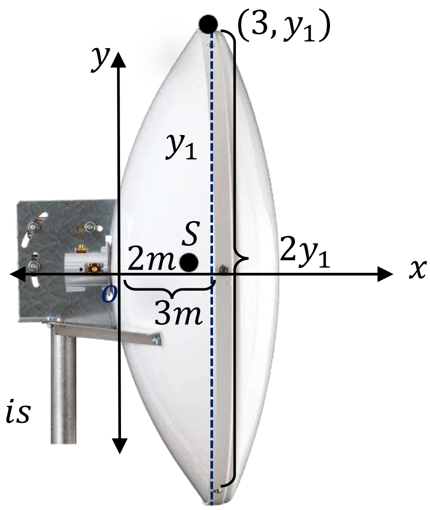
Since focus is 2m from the vertex $a = 2$

Equation of the parabola is $y^2 = 8x$

The point $(3, y_1)$ *lies on the parabola*

$$y_1^{\,2} = 8 \times 3$$

$$y_1 = \sqrt{8 \times 3} = \sqrt{4 \times 2 \times 3} = 2\sqrt{6}$$



The width of the antenna 3m from the vertex is

$$2y_1 = 4\sqrt{6}\text{m}.$$

Example 5.34 The equation $\boldsymbol{y = \frac{1}{32}x^2}$ ***models cross sections of parabolic mirrors that are used for solar energy. There is a heating tube located at the focus of each parabola; how high is this tube located above the vertex of the parabola?***

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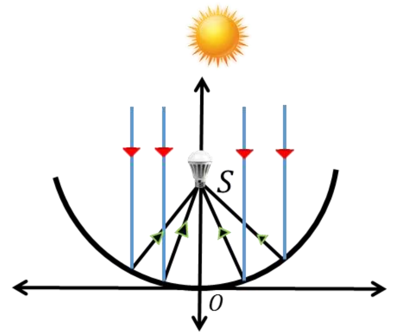


Equation of the parabola is $y = \frac{1}{32}x^2$

$$32y = x^2$$
$$x^2 = 32y$$

Compare with $x^2 = 4ay$

$$4a = 32 \Rightarrow a = \frac{32}{4} \; 8$$

$$\boxed{a = 8}$$

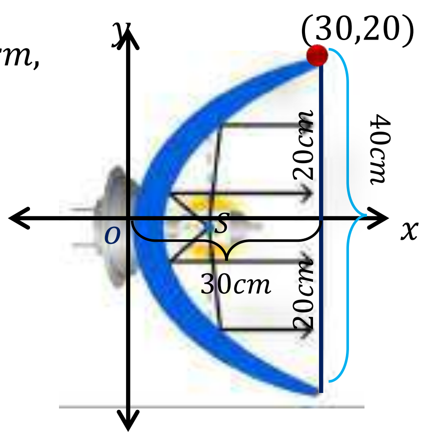
So the heating tube needs to be placed at focus .

Hence the heating tube needs to be placed 8 *units above the vertex of the parabola.*

Example* 5.35 *A search light has a parabolic reflector (has a cross section that. forms a 'bowl'). The parabolic bowl is* 40cm *wide from rim to rim and* 30cm *deep. The bulb is located at the focus

(*i*) *What is the equation of the parabola used for reflector*?

(*ii*) *How far from the vertex is the bulb to be placed so that the maximum distance covered*?



The equation of the parabola is $y^2 = 4\,ax$

Since the diameter is 40cm *and the depth is* 30cm,

The point $(30, 20)$ *lies on the parabola*

$$y^2 = 4ax \Rightarrow 20^2 = 4a(30)$$

$$400 = 120a \Rightarrow a = \frac{400}{120}\,\left(\frac{10}{3}\right) \Rightarrow a = \frac{10}{3}$$

$$\text{Sub } a = \frac{10}{3} \text{ in } y^2 = 4ax \Rightarrow y^2 = 4\left(\frac{10}{3}\right)x$$

The equation of the parabola is $y^2 = \frac{40}{3}x$

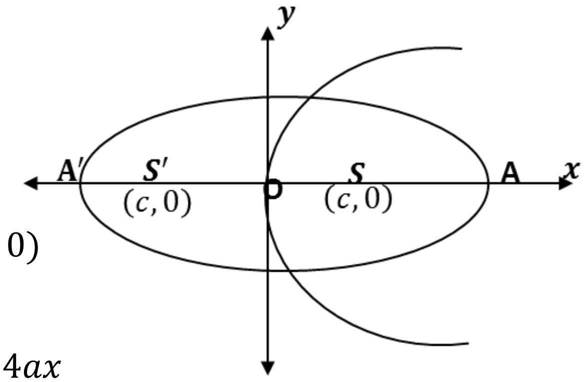
The bulb is at a distance of $\frac{10}{3}$ *cm from the vertex*

Example* 5.36 *An equation of the elliptical part of an optic lens system is $\boldsymbol{\frac{x^2}{16} + \frac{y^2}{9} = 1}$**. *The parabolic part of the system has a focus in common with the rightfocus of the ellipse. The vertex of the parabola is at the origin and theparabola opens to the right. Determine the equation of the parabola.***

In the given ellipse $a^2 = 16, b^2 = 9$

$$c^2 = a^2 - b^2 \quad \Rightarrow \quad c^2 = 16 - 9$$

$$c^2 = 7 \Rightarrow c = \pm\sqrt{7}$$



The foci are $S(\sqrt{7}, 0), S'(-\sqrt{7})$.

The focus of the parabola is $(\sqrt{7}, 0)$

$$\boxed{a = \sqrt{7}}$$

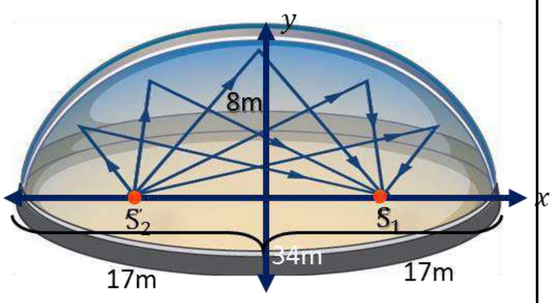
Equation of the parabola is $y^2 = 4ax$

Equation of the parabola is $y^2 = 4\sqrt{7}x$

Example 5.37 A room 34m long is constructed to be a whispering gallery. The room has an elliptical ceiling. If the maximumheight of the ceiling is 8m, determine where the foci are located.

The length of the semi major axis of the elliptical ceiling is 17m .

$$\boxed{a = 17}$$



The height of the semi minor axis is 8m

$$\boxed{b = 8}$$

$c^2 = a^2 - b^2 \Longrightarrow \ c = \sqrt{a^2 - b^2}$

$c = \sqrt{17^2 - 8^2} \Longrightarrow \ c = \sqrt{289 - 64}$

$c = \sqrt{225} \Longrightarrow \ c = 15$

For the elliptical ceiling the foci are located on either side about 15m *from the centre, along its major axis.*

Example 5.38 If the equation of the ellipse is $\dfrac{(x-11)^2}{484} + \dfrac{y^2}{64} = 1$ **(*x and y are measured centimeters*) *to the nearest centimeter, should the patient's kidney stone be placed so that the reflected sound hits the kidney stone*?**

The equation of the ellipse is $\dfrac{(x-11)^2}{484} + \dfrac{y^2}{64} = 1.$

20.49
2 | 420
  | 4
40 | 20
   | 00
404 | 2000
    | 1616
4089 | 38400
     | 36801
     | 1599

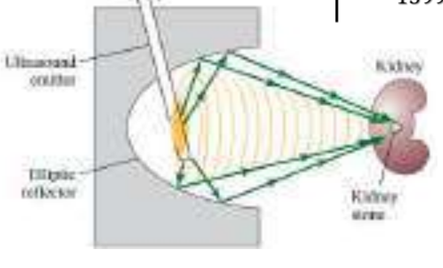
The origin of the sound wave and the kidney stone of patient should be at the foci in order to crush the stones.

$a^2 = 484$ *and* $b^2 = 64$

$c^2 = a^2 - b^2 \Longrightarrow \ c = \sqrt{a^2 - b^2}$

$c = \sqrt{484 - 64} \Longrightarrow \ c = \sqrt{420}$

$c \approx 20.5$



Therefore the patient's kidney stone should be placed 20.5 *cm from the Centre of the ellipse.*

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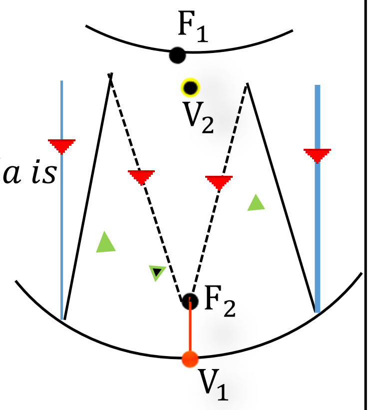
Example 5.40: ***Certain telescopes contain both parabolic mirror and a hyperbolic mirror. In the telescope shown in figure the parabola and hyperbola share focus F_1 which is 14m above the vertex of the parabola. The hypernola's second focus F_2 is 2m above the parabola's vertex. The vertex of the hyperbola and with the foci on the y − axis. Thenfind the equation of the hyperbola.***

Let V_1 be the vertex of the parabola and

V_2 be the vertex of the hyperbola.

$\overline{F_1F_2} = 14 - 2 = 12m, 2c = 12, c = 6$



The distance of centre to the vertex of the hyperbola is

$a = 6 - 1 = 5$

$b^2 = c^2 - a^2 \Longrightarrow b^2 = 6^2 - 5^2 \Longrightarrow b^2 = 36 - 25$

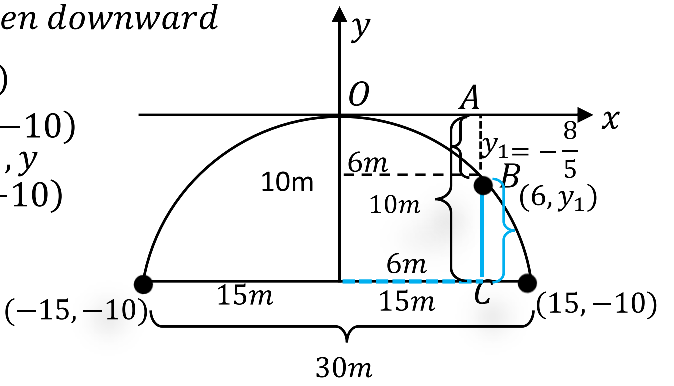
$b^2 = 11$

$\therefore$ *The equation of the hyperbola is* $\dfrac{y^2}{25} - \dfrac{x^2}{11} = 1.$

1. ***A bridge has a parabolic arch that is 10m high in the centre and 30m wide at the bottom. Find the height of the arch 6m from the centre, on either sides.***

A bridge of parabolic arc has open downward

$$x^2 = -4ay \ldots (1)$$



It passes through the point $(15, -10)$ (x , y)

$x^2 = -4ay \Longrightarrow 15^2 = -4a(-10)$

$225 = 40a \Longrightarrow a = \dfrac{225}{40}$

Sub $a = \dfrac{225}{40}$ in $x^2 = -4ay$

$x^2 = -4\left(\dfrac{225}{40}\right)y \Longrightarrow x^2 = -\dfrac{225}{10}y \Longrightarrow x^2 = \dfrac{-45}{2}y$

The point $(6, y_1)$ *lies on the parabola* $x^2 = \dfrac{-45}{2}y$

$6^2 = -\dfrac{45}{2}y_1 \Longrightarrow 36 = -\dfrac{45}{2}y_1$

$36 \times \dfrac{2}{45} = y_1 \Longrightarrow y_1 = \dfrac{-8}{5}$

$AB = \dfrac{8}{5}m$ *and* $AC = 10m$

$BC = AC - AB = 10 - \dfrac{8}{5} = \dfrac{50-8}{5} = \dfrac{42}{5}$

$BC = 8.4m$

The height of the bridge at the required place $= 8.4m$

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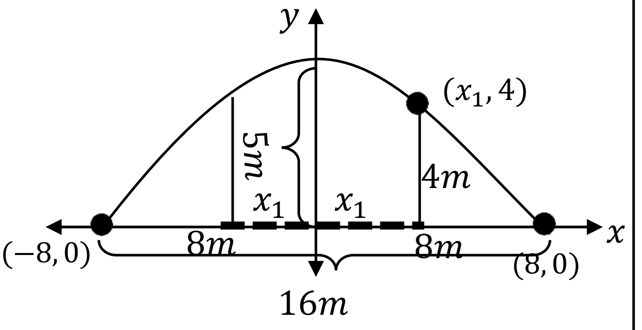
2. A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not the opening) is to be 16m, and the height at the edge of the road must be sufficient for a truck 4m high to clear if the highest point of the opening is to be 5m approximately. How wide must the opening be?



Here $a = 8$ *and* $b = 5$

Let x_1 *be the wide of the arch*

The equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The equation is $\frac{x^2}{8^2} + \frac{y^2}{5^2} = 1$

The point $(x_1, 4)$ *lies on the ellipse*

$$\frac{{x_1}^2}{64} + \frac{4^2}{25} = 1 \Rightarrow \frac{{x_1}^2}{64} + \frac{16}{25} = 1$$

$$\frac{{x_1}^2}{64} = 1 - \frac{16}{25} \Rightarrow \frac{{x_1}^2}{64} = \frac{25-16}{25} \Rightarrow {x_1}^2 = \frac{9}{25} \times 64$$

$$x_1 = \frac{3}{5} \times 8 \Rightarrow x_1 = \frac{24}{5}$$

wide of the arch $= 2x_1 = 2(4.8) = 9.6m$

3. At a water fountain, water attains a maximum height of 4m at horizontal distance of 0.5m from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of 0.75m from the point of origin.

Equation of a parabola: $(x-h)^2 = -4a(y-k)$

$$h = 0.5, k = 4$$

$$(x-0.5)^2 = -4a(y-4) \dots\dots (1)$$

It passes through origin $(0, 0)$ (x, y)

$$(0-0.5)^2 = -4a(0-4)$$

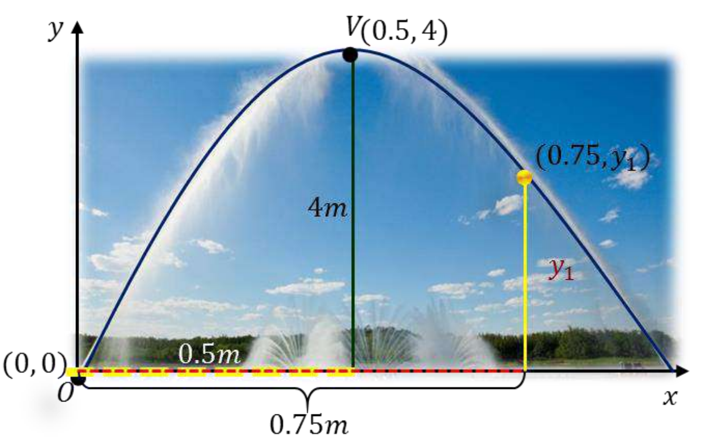
$$(-0.5)^2 = 16a \Rightarrow 16a = 0.25$$

$$a = \frac{0.25}{16}$$

$$(x-0.5)^2 = -4\left(\frac{0.25}{16}\right)(y-4)$$

(4 and 16 struck out, 16 replaced by 4)

$$(x-0.5)^2 = -\frac{0.25}{4}(y-4)$$



The point $(0.75, y_1)$ *lies on the parabola*

$$(x-0.5)^2 = -\frac{0.25}{4}(y-4) \Rightarrow (0.75-0.5)^2 = -\frac{0.25}{4}(y_1-4)$$

$$(0.25)^2 \times -\frac{4}{0.25} = y_1 - 4 \Rightarrow -0.25 \times 4 = y_1 - 4$$

$$-1 = y_1 - 4 \Rightarrow y_1 = 4-1 \Rightarrow y_1 = 3$$

Height of the water $= 3m$

4. An engineer designs a satellite dish with a parabolic cross section. The dish is 5m wide at the opening, and the focus is placed 1.2 m from the vertex

(a) Position a coordinate system with the origin at the vertex and the x − axis on the parabola's axis of symmetry and find an equation of the parabola.

(b) Find the depth of the satellite dish at the vertex.

Equation of a parabola: $y^2 = 4ax$

Here $a = 1.2m$

$$y^2 = 4(1.2)x \Rightarrow y^2 = 4.8x$$

Equation of a parabola: $y^2 = 4.8x$

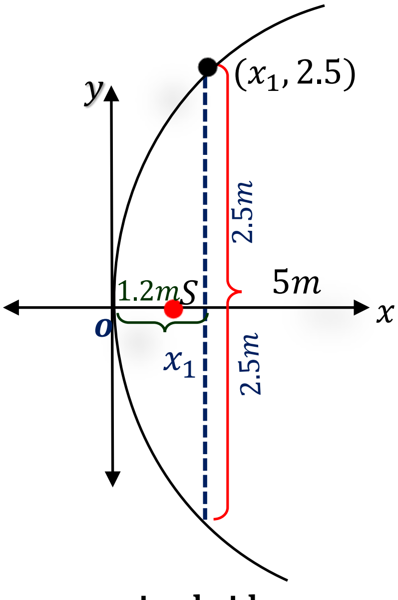
The point $(x_1, 2.5)$ *lies on the parabola*

$$y^2 = 4.8x \Rightarrow (2.5)^2 = 4.8x_1$$

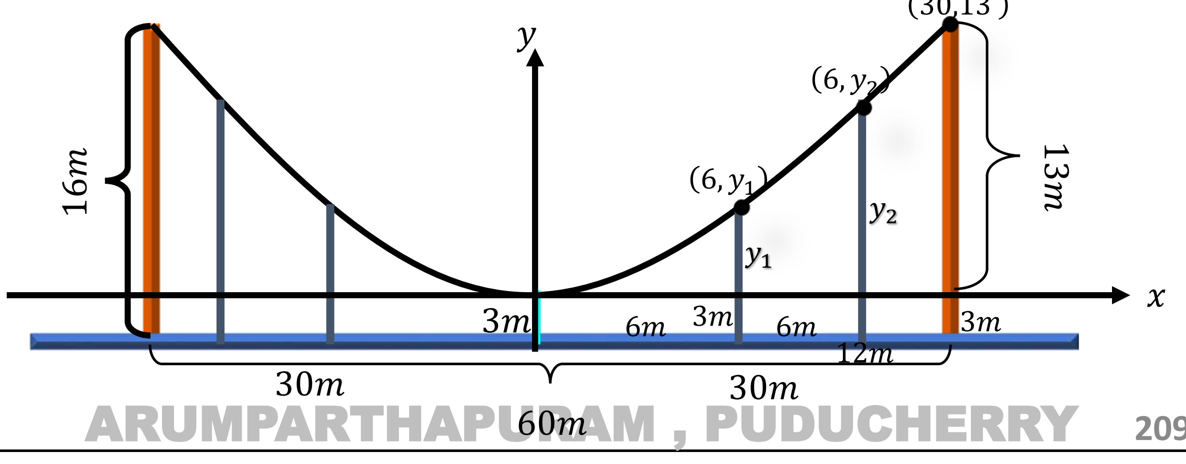
$$4.8x_1 = 6.25 \Rightarrow x_1 = \frac{6.25}{4.8} = \frac{625}{480} = \frac{125}{96}$$

$$x_1 = \frac{125}{96} = 1.3m$$

Depth of the satellite is $1.3m$



5.Parabolic cable of a 60m portion of the roadbed of a suspension bridge are positioned as shown below. Vertical Cables are to be spaced every 6m along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.



The equation of parabola is $x^2 = 4ay$

The point lies on the parabola $(30, 13)$

$$(30)^2 = 4a(13) \implies 900 = 52a \implies a = \frac{900}{52}$$

$$sub\ a = \frac{900}{52}\ in\ x^2 = 4ay$$

$$x^2 = 4\left(\frac{900}{52}\right)y \implies x^2 = \frac{900}{13}y$$

The equation is $x^2 = \frac{900}{13}y$

The point $(6, y_1)$ *lies on the parabola*

$$6^2 = \frac{900}{13}y_1 \implies \overset{4}{\cancel{36}} \times \frac{13}{\underset{100}{\cancel{900}}} = y_1 \implies y_1 = \frac{52}{100}$$

$$y_1 = 0.52$$

The length of thevertical cable 6m from the road$= 3 + 0.52 = 3.52m$

The point $(12, y_2)$ *also lies on the parabola*

$$12^2 = \frac{900}{13}y_2 \implies \overset{16}{\cancel{144}} \times \frac{13}{\underset{100}{\cancel{900}}} = y_2$$

$$y_2 = \frac{208}{100} \implies y_2 = 2.08$$

The length of the vertical cable 12m from the road $= 2.08 + 3 = 5.08m$

6. ***Cross section of a Nuclear cooling tower is in the shape of a hyperbola with equation*** $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$. ***The tower is 150m tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower.***

$$\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$$

(i) $(x_1, 50)$ *lies on the hyperbola*

$$\frac{{x_1}^2}{30^2} - \frac{50^2}{44^2} = 1 \implies \frac{{x_1}^2}{30^2} - \frac{2500}{1936} = 1$$

$$\frac{{x_1}^2}{30^2} = 1 + \frac{2500}{1936} \implies \frac{{x_1}^2}{30^2} = \frac{1936 + 2500}{1936}$$

$$\frac{{x_1}^2}{30^2} = \frac{4436}{1936} \implies {x_1}^2 = \frac{30^2 \times 4436}{44^2}$$

$$x_1 = \sqrt{\frac{30^2 \times 4436}{44^2}} \implies x_1 = \frac{30 \times \sqrt{4436}}{44}$$

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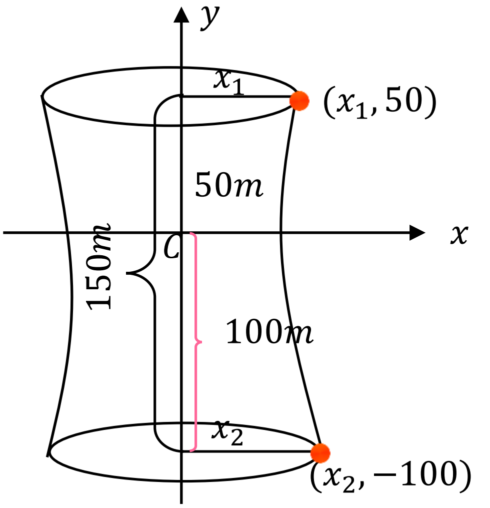
$$x_1 = \frac{15}{22} \times 66.6 \Rightarrow x_1 = \frac{15 \times 33.3}{11}$$

$$x_1 = \frac{499.5}{11} \Rightarrow x_1 = 45.41m$$



(ii)* $(x_2, -100)$ *lies on the hyperbola in

$$\frac{{x_2}^2}{30^2} - \frac{100^2}{44^2} = 1 \Rightarrow \frac{{x_2}^2}{30^2} - \frac{10000}{1936} = 1$$

$$\frac{{x_2}^2}{30^2} = 1 + \frac{10000}{1936} \Rightarrow \frac{{x_2}^2}{30^2} = \frac{1936 + 10000}{1936}$$

$$ {x_2}^2 = \frac{11936}{1936} \times 30^2 \Rightarrow x_2 = \sqrt{\frac{11936 \times 30^2}{44^2}}$$

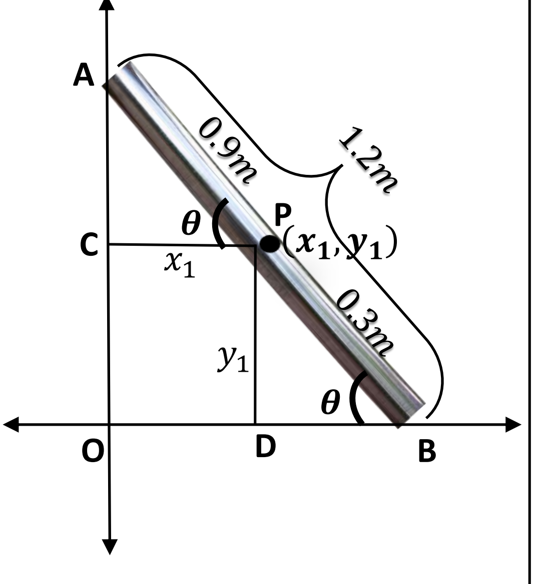
$$x_2 = \frac{30}{44}\sqrt{11936} \Rightarrow x_2 = 74.49m$$

$$Diameter\ for\ the\ top = 2x_1 = 2(45.41m) = 90.82m$$

$$Diameter\ for\ the\ bottom = 2x_2 = 2(74.49m) = 148.9m$$

7. A rod of length 1.2m moves with its ends always touching the coordinate axes. The locus of a point P on the rod, which is 0.3m from the end in contact with x − axis is an ellipse. Find the eccentricity.

The ladder AB = 1.2 m and point P on the ladder such that PB = 0.3m and AP = 0.9m



In a right angle triangle ACP

$$\cos\theta = \frac{adj}{hyp} \Rightarrow \cos\theta = \frac{x_1}{0.9}$$

In a right angle triangle PDB

$$\sin\theta = \frac{opp}{hyp} \Rightarrow \sin\theta = \frac{y_1}{0.3}$$

$$\cos\theta = \frac{x_1}{0.9} \text{ and } \sin\theta = \frac{y_1}{0.3}$$

$$cos^2\theta + sin^2\theta = 1$$

$$\left(\frac{x_1}{0.9}\right)^2 + \left(\frac{y_1}{0.3}\right)^2 = 1$$

$$\frac{{x_1}^2}{0.9^2} + \frac{{y_1}^2}{0.3^2} = 1$$

Compare with $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$

$a^2 = 0.9^2, b^2 = 0.3^2$

$a = 0.9$

To Find eccentricity (e) $e = \frac{\sqrt{a^2 - b^2}}{a}$

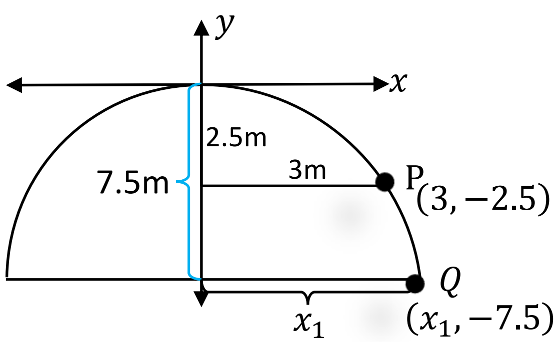
$e = \frac{\sqrt{0.9^2 - 0.3^2}}{0.9} \Rightarrow e = \frac{\sqrt{0.81 - 0.09}}{0.9} \Rightarrow e = \frac{\sqrt{0.72}}{0.9} \Rightarrow e = \frac{\sqrt{\frac{72}{100}}}{0.9}$

$e = \frac{\sqrt{\frac{9 \times 8}{100}}}{0.9} \Rightarrow e = \frac{\frac{3\sqrt{8}}{10}}{0.9} \Rightarrow e = \frac{3\sqrt{8}}{10} \times \frac{1}{0.9} \Rightarrow e = \frac{3 \times 2\sqrt{2}}{9} \Rightarrow \boxed{e = \frac{2\sqrt{2}}{3}}$

8. Assume that water issuing from the end of a horizontal pipe, 7.5m above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position 2.5m below the line of the pipe, the flow of water has curved outward 3m beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground.

The equation of the parabola as open downward $x^2 = -4ay$

It passes through the point $P(3, -2.5)$



$3^2 = -4a(-2.5)$

$9 = 10a \Rightarrow a = \frac{9}{10}$

Sub $a = \frac{9}{10}$ in $x^2 = -4ay$

$x^2 = -4\left(\frac{9}{10}\right)y$ (4 cancelled to 2, 10 cancelled to 5)

$\therefore$ The equation is $x^2 = -\frac{18}{5}y$

Let x_1 be the distance between the Vertical line and water strike the ground

The point $Q(x_1, -7.5)$ lies on the parabola

$x^2 = -\frac{18}{5}y \Rightarrow x_1^2 = -\frac{18}{5}(-7.5)$

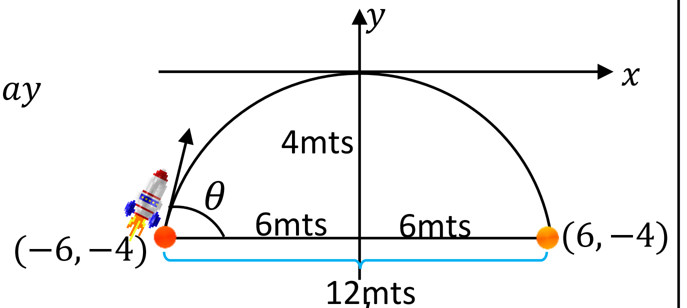
$x_1^2 = \frac{18}{5} \times 7.5$ (7.5 ÷ 5 = 1.5) $\Rightarrow x_1^2 = 18 \times 1.5$

$x_1^2 = 9 \times 2 \times 1.5 \Rightarrow x_1^2 = 9 \times 3$

$x_1 = \sqrt{9 \times 3} \Rightarrow x_1 = 3\sqrt{3}m$

$\therefore$ The water strikes the ground $3\sqrt{3}m$ beyond the vertical line

9. On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of 4mts when it is 6mts away from the point of projection. Finally it reaches the ground 12mts away from the starting point. Find the angle of projection



$$Slope = \frac{dy}{dx} = tan\theta$$

The equation of parabola is $x^2 = -4ay$

It passes through the point $(\underset{x,y}{6,-4})$

$$x^2 = -4ay \implies 6^2 = -4a(-4)$$

$$16a = 36 \implies a = \frac{36}{16} \implies a = \frac{9}{4}$$

Sub $a = \frac{9}{4}$ in $x^2 = -4ay$

$$x^2 = -4\left(\frac{9}{4}\right)y$$

The equation is $x^2 = -9y$

$$x^2 = -9y$$

$d.w.r.to.x$

$$2x = -9\frac{dy}{dx} \implies \frac{dy}{dx} = -\frac{2x}{9}$$

$$\frac{dy}{dx} = -\frac{2x}{9}$$

Find the slope at $(-6,-4)$ $\frac{dy}{dx} = \frac{-2(-6)}{9} = \frac{12}{9} = \frac{4}{3}$

$$\frac{dy}{dx} = \frac{4}{3} \quad i.e \ \tan\theta = \frac{4}{3}$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

The angle of projection $\tan^{-1}\left(\frac{4}{3}\right)$

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VECTOR ALGEBRA

EXERCISE : 6.1

Example 6.1:* *Prove by vector method that $\cos A = \frac{b^2+c^2-a^2}{2bc}$

$\overrightarrow{AB} = \vec{c}, \overrightarrow{BC} = \vec{a}$ and $\overrightarrow{CA} = \vec{b}$

By triangular law

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0} \Rightarrow \vec{c} + \vec{a} + \vec{b} = \vec{0}$$

$$\vec{b} + \vec{c} = -\vec{a}$$

squaring on both sides

$$\left(\vec{b} + \vec{c}\right)^2 = (-\vec{a})^2 \Rightarrow \vec{b}^2 + \vec{c}^2 + 2\left(\vec{b}.\vec{c}\right) = \vec{a}^2$$

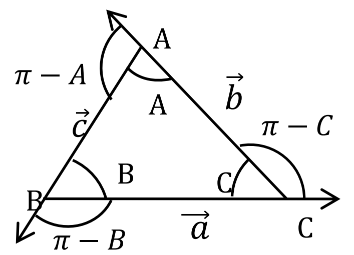
$$\vec{b}^2 + \vec{c}^2 + 2\left|\vec{b}\right|\left|\vec{c}\right|\cos(\pi - A) = \vec{a}^2$$

$$b^2 + c^2 + 2bc(-\cos A) = a^2$$

$$b^2 + c^2 - 2bc\cos A = a^2$$

$$b^2 + c^2 - a^2 = 2bc\cos A$$

$$\boxed{\cos A = \frac{b^2 + c^2 - a^2}{2bc}}$$



$\boxed{\cos(180° - \theta) = -\cos\theta}$

(iii) Prove by vector method that $\cos C = \frac{a^2+b^2-c^2}{2ab}$

$\overrightarrow{AB} = \vec{c}, \overrightarrow{BC} = \vec{a}$ and $\overrightarrow{CA} = \vec{b}$

By triangular law

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0} \Rightarrow \vec{c} + \vec{a} + \vec{b} = \vec{0}$$

$$\vec{a} + \vec{b} = -\vec{c}$$

squaring on both side

$$\left(\vec{a} + \vec{b}\right)^2 = (-\vec{c})^2 \Rightarrow \vec{a}^2 + \vec{b}^2 + 2\left(\vec{a}.\vec{b}\right) = \vec{c}^2$$

$$a^2 + b^2 + 2|\vec{a}|\left|\vec{b}\right|\cos(\pi - C) = c^2 \Rightarrow a^2 + b^2 + 2ab(-\cos C) = c^2$$

$$a^2 + b^2 - 2ab\cos C = c^2 \Rightarrow a^2 + b^2 - c^2 = 2ab\cos C$$

$$\boxed{\cos C = \frac{a^2 + b^2 - c^2}{2ab}}$$

Example 6.2: Prove by vector method that a = b cosC + c cosB

$\overrightarrow{AB} = \vec{c}, \overrightarrow{BC} = \vec{a}$ *and* $\overrightarrow{CA} = \vec{b}$

By triangular law

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0} \Rightarrow \vec{c} + \vec{a} + \vec{b} = \vec{0}$$

$$\vec{a} = -\vec{b} - \vec{c}$$

Dot multiplying both side by $\vec{a}$

$$\vec{a}\,.\,\vec{a} = -\vec{a}.\,\vec{b} - \vec{a}.\,\vec{c}$$

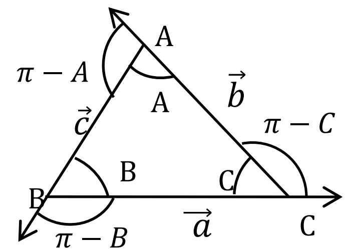
$$\vec{a}^2 = -|\vec{a}||\vec{b}|\,cos(\pi - C) - |\vec{a}||\vec{c}|\,cos(\pi - B)$$

$$a^2 = -ab(-cosC) - ac(-cosB)$$

$$a^2 = abcosC + ac\,cosB$$

$$\div a$$

$$a = b\,cosC + c\,cosB$$



$cos(\pi - \theta) = -cos\theta$

(ii) Prove by vector method that b = c cosA + a cosC

From the diagram

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$$

$$\vec{c} + \vec{a} + \vec{b} = \vec{0}$$

$$\vec{b} = -\vec{a} - \vec{c}$$

Dot multiplying both side by $\vec{b}$

$$\vec{b}\,.\,\vec{b} = -\vec{a}.\,\vec{b} - \vec{b}.\,\vec{c}$$

$$\vec{b}^2 = -|\vec{a}||\vec{b}|\,cos(\pi - C) - |\vec{b}||\vec{c}|\,cos(\pi - A)$$

$$b^2 = -ab(-cosC) - bc(-cosA)$$

$$b^2 = abcosC + bccosA$$

$$\div b$$

$$b = a\,cosC + c\,cosA$$

(iii) Prove by vector method that c = acosB + bcosA

$$\overrightarrow{AB} = \vec{c}\,, \overrightarrow{BC} = \vec{a} \text{ and } \overrightarrow{CA} = \vec{b}$$

From the diagram

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$$

$$\vec{c} + \vec{a} + \vec{b} = \vec{0}$$

$$\vec{c} = -\vec{a} - \vec{b}$$

Dot multiplying both side by $\vec{c}$

$$\vec{c}\,.\,\vec{c} = -\vec{a}.\vec{c} - \vec{b}.\vec{c}$$

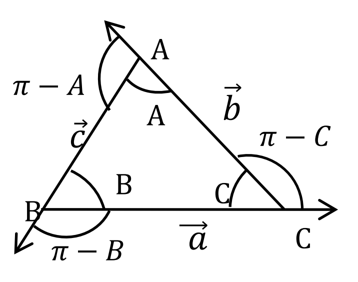
$$\vec{c}^2 = -|\vec{a}||\vec{c}|\,cos(\pi - B) - |\vec{b}||\vec{c}|cos(\pi - A)$$

$$c^2 = -ac(-cosB) - bc(-cosA)$$

$$c^2 = ac\,cosB + bc\,cosA$$

$$\div c$$

$$c = a\,cosB + b\,cosA$$



Example 6.4: With usual notation prove that $\frac{a}{sin\,A} = \frac{b}{sin\,B} = \frac{c}{sin\,C}$

Let $\overrightarrow{AB} = \vec{c}, \overrightarrow{BC} = \vec{a}, \overrightarrow{CA} = \vec{b}$

By Area property of triangles

$$\frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}|\vec{b}\times\vec{c}| = \frac{1}{2}|\vec{c}\times\vec{a}| \Longrightarrow |\vec{a}\times\vec{b}| = |\vec{b}\times\vec{c}| = |\vec{c}\times\vec{a}|$$

$$\div \frac{1}{2}$$

$$|\vec{a}||\vec{b}|sin(\pi - C) = |\vec{b}||\vec{c}|sin(\pi - A) = |\vec{c}||\vec{a}|\, sin(\pi - B)$$

$$absinC = bcsinA = casinB \Longrightarrow \frac{absinC}{abc} = \frac{bcsinA}{abc} = \frac{casinB}{abc}$$

$$\div\, abc$$

$$\frac{sinC}{c} = \frac{sinA}{a} = \frac{sinB}{b}$$

Taking reciprocal

$$\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$$

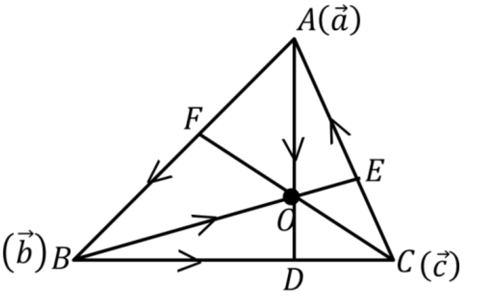
***Example 6.7**: Altitudes of a triangles are concurrent – prove by vector method.*

let the position vectors of A, B, C be $\vec{a}$, $\vec{b}$, $\vec{c}$.

$$\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}$$

Let ABC be a triangle and let AD, BE be its two altitudes intersecting at O.

We shall prove CF is the third altitude also intersecting at O.



$$\overrightarrow{AD} \perp \overrightarrow{BC}$$

$$\overrightarrow{AD}\,.\,\overrightarrow{BC} = 0 \Longrightarrow \overrightarrow{OA}\,.\,\overrightarrow{BC} = 0$$

$$\overrightarrow{OA}\,.\left(\overrightarrow{OC} - \overrightarrow{OB}\right) = 0 \Longrightarrow \vec{a}\,.\left(\vec{c} - \vec{b}\right) = 0$$

$$\vec{a}\,.\,\vec{c} - \vec{a}\,.\,\vec{b} = 0 \quad \ldots (1)$$

$$\overrightarrow{BE} \perp \overrightarrow{CA}$$

$$\overrightarrow{BE}\,.\,\overrightarrow{CA} = 0 \Longrightarrow \overrightarrow{OB}\,.\,\overrightarrow{CA} = 0$$

$$\overrightarrow{OB}\,.\left(\overrightarrow{OA} - \overrightarrow{OC}\right) = 0 \Longrightarrow \vec{b}\,.(\vec{a} - \vec{c}) = 0$$

$$\vec{b}\,.\,\vec{a} - \vec{b}\,.\,\vec{c} = 0 \quad \ldots (2)$$

Adding (1) and (2)

$$\vec{a}\,.\,\vec{c} - \vec{a}\,.\,\vec{b} = 0$$

$$\vec{b}\,.\,\vec{a} - \vec{b}\,.\,\vec{c} = 0$$

$$\vec{a}\,.\,\vec{c} - \vec{b}\,.\,\vec{c} = 0$$

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$$(\vec{a}-\vec{b}).\vec{c}=0$$

$$(\overrightarrow{OA}-\overrightarrow{OB}).\overrightarrow{OC}=0$$

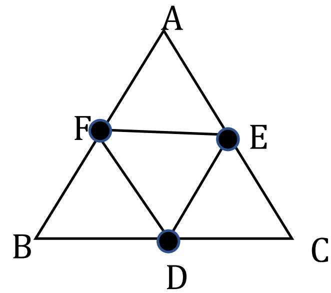
$$\overrightarrow{BA}\,.\,\overrightarrow{OC}=0 \Longrightarrow \overrightarrow{BA}\,.\,\overrightarrow{CF}=0$$

$$\overrightarrow{BA}\perp\overrightarrow{CF}$$

Hence the three altitudes are concurrent.

Example 6.8

In triangle, ABC the points D, E, F are the midpoints of the sides , BC, CA and AB respectively. Using vector method, show that the area of ΔDEF is equal to $\frac{1}{4}$(area of ΔABC).



In triangle , ABC consider O as the origin

Let D, E, F are the midpoints of BC, CA, AB

$$\overrightarrow{OD}=\frac{\overrightarrow{OB}+\overrightarrow{OC}}{2},\quad \overrightarrow{OE}=\frac{\overrightarrow{OA}+\overrightarrow{OC}}{2},\quad \overrightarrow{OF}=\frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}$$

$$Area\ of\ \Delta DEF=\frac{1}{2}\left|\overrightarrow{DE}\times\overrightarrow{DF}\right|=\frac{1}{2}\left|(\overrightarrow{OE}-\overrightarrow{OD})\times(\overrightarrow{OF}-\overrightarrow{OD})\right|$$

$$=\frac{1}{2}\left|\left(\frac{\overrightarrow{OA}+\overrightarrow{OC}}{2}-\frac{\overrightarrow{OB}+\overrightarrow{OC}}{2}\right)\times\left(\frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}-\frac{\overrightarrow{OB}+\overrightarrow{OC}}{2}\right)\right|$$

$$=\frac{1}{2}\left|\left(\frac{\overrightarrow{OA}+\cancel{\overrightarrow{OC}}-\overrightarrow{OB}-\cancel{\overrightarrow{OC}}}{2}\right)\times\left(\frac{\overrightarrow{OA}+\cancel{\overrightarrow{OB}}-\cancel{\overrightarrow{OB}}-\overrightarrow{OC}}{2}\right)\right|$$

$$=\frac{1}{2}\left|\left(\frac{\overrightarrow{OA}-\overrightarrow{OB}}{2}\right)\times\left(\frac{\overrightarrow{OA}-\overrightarrow{OC}}{2}\right)\right|=\frac{1}{2}\left|\left(\frac{\overrightarrow{BA}}{2}\right)\times\left(\frac{\overrightarrow{CA}}{2}\right)\right|$$

$$=\frac{1}{4}\left|\frac{\overrightarrow{BA}\times\overrightarrow{CA}}{2}\right|=\frac{1}{4}\left(\frac{1}{2}\left|\overrightarrow{BA}\times\overrightarrow{CA}\right|\right)$$

$$=\frac{1}{4}(Area\ of\ a\ \Delta ABC)$$

Example 6.9: The constant forces $2\hat{\mathbf{i}}+5\hat{\mathbf{j}}+6\hat{\mathbf{k}}$, $-\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}}$ act on a particle which is displaced from position $(4,-3,-2)$ to position $(6,1,-3)$. Find the work done

Let $\overrightarrow{F_1}=2\hat{\imath}+5\hat{\jmath}+6\hat{k}$, $\overrightarrow{F_2}=-\hat{\imath}-2\hat{\jmath}-\hat{k}$

$$\vec{F}=\overrightarrow{F_1}+\overrightarrow{F_2}$$

$$=2\hat{\imath}+5\hat{\jmath}+6\hat{k}-\hat{\imath}-2\hat{\jmath}-\hat{k}$$

$$\vec{F}=\hat{\imath}+3\hat{\jmath}+5\hat{k}$$

Let $\overrightarrow{OA}=4\hat{\imath}-3\hat{\jmath}-2\hat{k}$, $\overrightarrow{OB}=6\hat{\imath}+\hat{\jmath}-3\hat{k}$

$\vec{d} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\vec{d} = 6\hat{i} + \hat{j} - 3\hat{k} - 4\hat{i} + 3\hat{j} + 2\hat{k}$

$\vec{d} = 2\hat{i} + 4\hat{j} - \hat{k}$

Work done = $\vec{F}.\vec{d} = (\hat{i} + 3\hat{j} + 5\hat{k}).(2\hat{i} + 4\hat{j} - \hat{k})$

$= 2 + 12 - 5 = 9\ units$

Example 6.9: A partical is acted upon by the forces $3\hat{i} - 2\hat{j} + 2\hat{k}$, $2\hat{i} + \hat{j} - \hat{k}$ *is displaced* from the point $(1, 3, -1)$ to the point *to* $(4, -1, \lambda)$. *if the work done by the forces* 16 *units find the value of* λ.

Let $\overrightarrow{F_1} = 3\hat{i} - 2\hat{j} + 2\hat{k}, \overrightarrow{F_2} = 2\hat{i} + \hat{j} - \hat{k}$

$\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2}$

$= 3\hat{i} - 2\hat{j} + 2\hat{k} + \vec{i} + 2\hat{i} + \hat{j} - \hat{k}$

$\vec{F} = 5\hat{i} - \hat{j} + \hat{k}$

Let $\overrightarrow{OA} = \hat{i} + 3\hat{j} - \hat{k}, \overrightarrow{OB} = 4\hat{i} - \hat{j} + \lambda\hat{k}$

$\vec{d} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 4\hat{i} - \hat{j} + \lambda\hat{k} - \hat{i} - 3\hat{j} + \hat{k}$

$\vec{d} = 3\hat{i} - 4\hat{j} + (\lambda + 1)\hat{k}$

Work done = 16 units

$\vec{F}.\vec{d} = 16 \Rightarrow (5\hat{i} - \hat{j} + \hat{k}).3\hat{i} - 4\hat{j} + (\lambda + 1)\hat{k} = 16$

$15 + 4 + \lambda + 1 = 16 \Rightarrow 20 + \lambda = 16 \Rightarrow \lambda = 16 - 20$

$\boxed{\lambda = -4}$

Example 6.11: Find the magnitude and direction cosines of the moment about the point $(2, 0, -1)$ of a force $2\hat{i} + \hat{j} - \hat{k}$ whose line of action passes through the origin.

$\vec{F} = 2\hat{i} + \hat{j} - \hat{k}, \ \overrightarrow{OA} = 2\hat{i} + 0\hat{j} - \hat{k}, \ \overrightarrow{OB} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$\vec{r} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 0\hat{i} + 0\hat{j} + 0\hat{k} - 2\hat{i} + 0\hat{j} + \hat{k}$

$\vec{r} = -2\hat{i} + \hat{k}$

$\boxed{Moment\ of\ a\ force = \vec{r} \times \vec{F}}$

$$\vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \hat{i}(0 - 1) - \hat{j}(2 - 2) + \hat{k}(-2 - 0)$$

$\vec{r} \times \vec{F} = -\hat{i} - 0\hat{j} - 2\hat{k}$

$\boxed{\text{Magnitude of moment} = |\vec{r} \times \vec{F}|}$

$$= \sqrt{(-1)^2 + 0^2 + (-2)^2}$$

$$= \sqrt{1+4} = \sqrt{5}$$

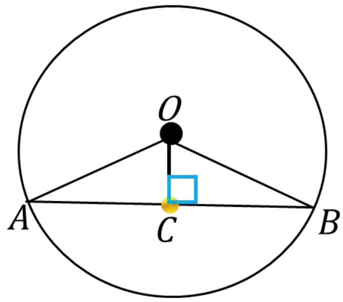
$$Direction\ cosines = \left(\frac{-1}{\sqrt{5}}, \frac{0}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right) = \left(\frac{-1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}\right)$$

1. *Prove by vector method that if a line is drawn from the centre of a circle to the midpoint of a chord, then the line is perpendicular to the chord.*

Let O be a centre of acircle and AB be a chord

Let C be the midpoint of the sides AB

$$\overrightarrow{OC} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$$



$OA = OB$

To prove $\overrightarrow{OC} \perp \overrightarrow{AB}$

$$\overrightarrow{OC}.\overrightarrow{AB} = \overrightarrow{OC}.\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$

$$= \frac{\left(\overrightarrow{OA} + \overrightarrow{OB}\right)}{2}.\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$

$$= \frac{\left(\overrightarrow{OB} + \overrightarrow{OA}\right).\left(\overrightarrow{OB} - \overrightarrow{OA}\right)}{2}$$

$$= \frac{\overrightarrow{OB}^2 - \overrightarrow{OA}^2}{2} = \frac{OB^2 - OA^2}{2} = \frac{OB^2 - OB^2}{2}$$

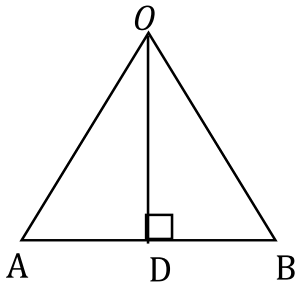
$$\therefore \overrightarrow{OC} \perp \overrightarrow{AB} \Rightarrow \overrightarrow{OC}.\overrightarrow{AB} = 0$$

2. *Prove by vector method that the median to the base of an isosceles triangle is perpendicular to the base.*

Let O be the origin

Let D is the mid − point of the side AB

$$\overrightarrow{OD} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$$



$OA = OB$

To prove $\overrightarrow{OD} \perp \overrightarrow{AB}$

$$\overrightarrow{OD}.\overrightarrow{AB} = \overrightarrow{OD}.\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$

$$= \frac{\left(\overrightarrow{OA} + \overrightarrow{OB}\right)}{2}.\left(\overrightarrow{OB} - \overrightarrow{OA}\right) = \frac{\left(\overrightarrow{OB} + \overrightarrow{OA}\right).\left(\overrightarrow{OB} - \overrightarrow{OA}\right)}{2} = \frac{\overrightarrow{OB}^2 - \overrightarrow{OA}^2}{2}$$

$$\overrightarrow{OD}.\overrightarrow{AB} = \frac{OB^2 - OB^2}{2} \Rightarrow \overrightarrow{OD}.\overrightarrow{AB} = 0 \Rightarrow \therefore \overrightarrow{OD} \perp \overrightarrow{AB}$$

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3. Angle in a semi-circle is a right angle. Prove by vector method

Let AB be the diameter of the circle with centre O.

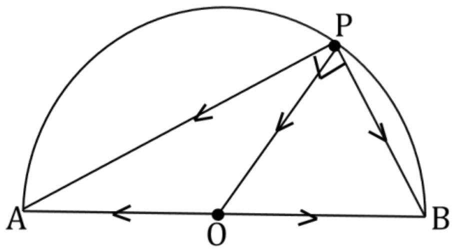
Let P be any point on the semi-circle

$OA = OB = OP = radii$

$\overrightarrow{PA} = \overrightarrow{PO} + \overrightarrow{OA}$

$\overrightarrow{PB} = \overrightarrow{PO} + \overrightarrow{OB}$

$\overrightarrow{PB} = \overrightarrow{PO} - \overrightarrow{OA}$



$\overrightarrow{OB} = -\overrightarrow{OA}$

[Equal magnitude but opposite in direction]

$\overrightarrow{PA} . \overrightarrow{PB} = (\overrightarrow{PO} + \overrightarrow{OA}).(\overrightarrow{PO} - \overrightarrow{OA})$

$= \overrightarrow{PO}^2 - \overrightarrow{OA}^2 = PO^2 - OA^2$

$= OA^2 - OA^2 = 0$

$\overrightarrow{PA} . \overrightarrow{PB} = 0$ i.e $\overrightarrow{PA} \perp \overrightarrow{PB}$

$\angle P = 90°$

Hence angle in a semi−circle is a right angle

4. Diagonals of a rhombus are at right angles. Prove by vector methods

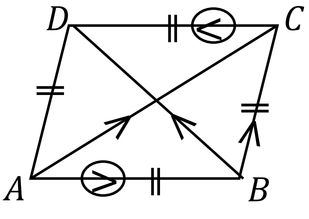
In a rhombus AB = BC = CD = DA

Let ABCD be a rhombus.

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

$\overrightarrow{BD} = \overrightarrow{BC} - \overrightarrow{AB}$



$\overrightarrow{CD} = -\overrightarrow{AB}$

$\overrightarrow{CD}$ and $\overrightarrow{AB}$ are equal in magnitude but opposite in direction

$\overrightarrow{AC} . \overrightarrow{BD} = (\overrightarrow{AB} + \overrightarrow{BC}).(\overrightarrow{BC} - \overrightarrow{AB})$

$= (\overrightarrow{BC} + \overrightarrow{AB}).(\overrightarrow{BC} - \overrightarrow{AB})$

$= \overrightarrow{BC}^2 - \overrightarrow{AB}^2 = BC^2 - AB^2$

$= BC^2 - BC^2 = 0 \qquad \therefore AB = BC$

$\overrightarrow{AC} . \overrightarrow{BD} = 0$ then $\overrightarrow{AC} \perp \overrightarrow{BD}$

Hence diagonals of a rhombus are at right angle

5. If the diagonals of a parallelogram are equal then it is a rectangle. Prove by vector method

Let ABCD be a parallelogram

Given that : AC = BD

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$|\overrightarrow{AC}| = |\overrightarrow{AB} + \overrightarrow{BC}|$ $\qquad AC = |\overrightarrow{AB} + \overrightarrow{BC}|$

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

$\overrightarrow{BD} = \overrightarrow{BC} - \overrightarrow{AB} \quad \therefore -\overrightarrow{AB} = \overrightarrow{CD}$

$|\vec{BD}| = |\vec{BC} - \vec{AB}|$

$BD = |\vec{BC} - \vec{AB}|$

$AC = BD$

$|\vec{AB} + \vec{BC}| = |\vec{BC} - \vec{AB}|$

Squaring on both sides

$|\vec{AB} + \vec{BC}|^2 = |\vec{BC} - \vec{AB}|^2$

$(\vec{AB} + \vec{BC})^2 = (\vec{BC} - \vec{AB})^2$

$\vec{AB}^2 + \vec{BC}^2 + 2(\vec{AB}.\vec{BC}) = \vec{BC}^2 + \vec{AB}^2 - 2(\vec{AB}.\vec{BC})$

$2(\vec{AB}.\vec{BC}) = -2(\vec{AB}.\vec{BC})$

$2(\vec{AB}.\vec{BC}) + 2(\vec{AB}.\vec{BC}) = 0$

$4(\vec{AB}.\vec{BC}) = 0 \Rightarrow \vec{AB}.\vec{BC} = 0 \Rightarrow \vec{AB} \perp \vec{BC}$

Hence the parallelogram is a rectangle

6. Prove that the area of a quadrilateral ABCD is $\frac{1}{2}|\vec{AC} \times \vec{BD}|$ where AC and BD are its diagonals.

vector area of a Quadrilateral ABCD = vector area of a ΔABC + vector area of a ΔACD

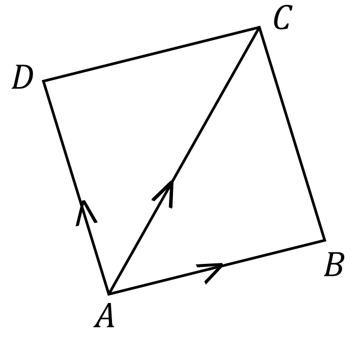
$= \frac{1}{2}(\vec{AB} \times \vec{AC}) + \frac{1}{2}(\vec{AC} \times \vec{AD})$

$= \frac{1}{2}(-\vec{AC} \times \vec{AB}) + \frac{1}{2}(\vec{AC} \times \vec{AD})$

$= \frac{1}{2}\vec{AC} \times (-\vec{AB} + \vec{AD})$

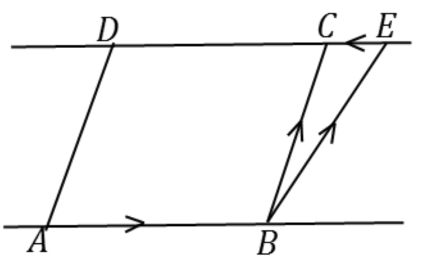
$= \frac{1}{2}\vec{AC} \times (\vec{BA} + \vec{AD})$

$= \frac{1}{2}(\vec{AC} \times \vec{BD})$



Area of a Quadrilateral ABCD $= \frac{1}{2}|\vec{AC} \times \vec{BD}|$

7. Prove by vector method that the parallelograms on the same base and between the same parallels are equal in area.



Area of a parallelogram ABCD $= |\vec{AB} \times \vec{BC}|$

$= |\vec{AB} \times (\vec{BE} + \vec{EC})|$

$= |\vec{AB} \times \vec{BE} + \vec{AB} \times \vec{EC}|$

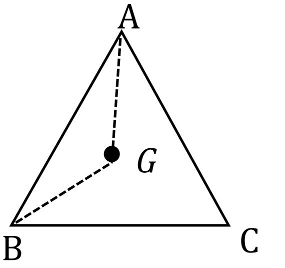
$= |\vec{AB} \times \vec{BE} + \vec{0}| \quad \because \vec{AB} \parallel \vec{EC}$

$= |\overrightarrow{AB} \times \overrightarrow{BE}|$

$=$ Area of a parallelogram ABED

8. *If G is the centroid of a ABC Δ, prove that (area of ΔGAB) = (area of ΔGBC) = (area of GCAΔ) = $\frac{1}{3}$ (area of Δ ABC).*

To prove : area of ΔGAB = area of ΔGBC = area of ΔGCA $= \frac{1}{3}$ *area of ΔABC*



$$\overrightarrow{OG} = \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}$$

$$\text{area of } \Delta GAB = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AG}| = \frac{1}{2}|\overrightarrow{AB} \times (\overrightarrow{OG} - \overrightarrow{OA})|$$

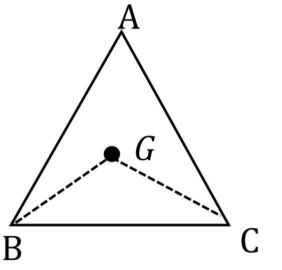
$$= \frac{1}{2}\left|\overrightarrow{AB} \times \left(\frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3} - \overrightarrow{OA}\right)\right|$$

$$= \frac{1}{2}\left|\overrightarrow{AB} \times \left(\frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} - 3\overrightarrow{OA}}{3}\right)\right| = \frac{1}{2}\left|\overrightarrow{AB} \times \left(\frac{\overrightarrow{OB} + \overrightarrow{OC} - 2\overrightarrow{OA}}{3}\right)\right|$$

$$= \frac{1}{2}\left|\overrightarrow{AB} \times \left(\frac{\overrightarrow{OB} - \overrightarrow{OA} + \overrightarrow{OC} - \overrightarrow{OA}}{3}\right)\right| = \frac{1}{2}\left|\overrightarrow{AB} \times \left(\frac{\overrightarrow{AB} + \overrightarrow{AC}}{3}\right)\right|$$

$$= \frac{1}{6}|\overrightarrow{AB} \times (\overrightarrow{AB} + \overrightarrow{AC})| = \frac{1}{6}|\overrightarrow{AB} \times \overrightarrow{AB} + \overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{6}|\vec{0} + \overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{3} \times \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$



$$\text{area of } \Delta GAB = \frac{1}{3}(\text{area of } \Delta ABC) \quad \ldots (1)$$

$$\text{area of } \Delta GBC = \frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BG}| = \frac{1}{2}|\overrightarrow{BC} \times (\overrightarrow{OG} - \overrightarrow{OB})|$$

$$= \frac{1}{2}\left|\overrightarrow{BC} \times \left(\frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3} - \overrightarrow{OB}\right)\right| = \frac{1}{2}\left|\overrightarrow{BC} \times \left(\frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} - 3\overrightarrow{OB}}{3}\right)\right|$$

$$= \frac{1}{2}\left|\overrightarrow{BC} \times \left(\frac{\overrightarrow{OA} + \overrightarrow{OC} - 2\overrightarrow{OB}}{3}\right)\right| = \frac{1}{2}\left|\overrightarrow{BC} \times \left(\frac{\overrightarrow{OA} - \overrightarrow{OB} + \overrightarrow{OC} - \overrightarrow{OB}}{3}\right)\right|$$

$$= \frac{1}{2}\left|\overrightarrow{BC} \times \left(\frac{\overrightarrow{OA} - \overrightarrow{OB} + \overrightarrow{OC} - \overrightarrow{OB}}{3}\right)\right|$$

$$= \frac{1}{2}\left|\overrightarrow{BC} \times \left(\frac{\overrightarrow{BA} + \overrightarrow{BC}}{3}\right)\right| = \frac{1}{6}|\overrightarrow{BC} \times (\overrightarrow{BA} + \overrightarrow{BC})|$$

$$= \frac{1}{6}|\overrightarrow{BC} \times \overrightarrow{BA} + \overrightarrow{BC} \times \overrightarrow{BC}| = \frac{1}{6}|\overrightarrow{BC} \times \overrightarrow{BA} + \vec{0}| = \frac{1}{6}|\overrightarrow{BC} \times \overrightarrow{BA}|$$

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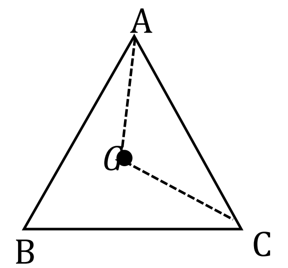


$$= \frac{1}{3} \times \frac{1}{2} \left|\overrightarrow{BC} \times \overrightarrow{BA}\right|$$

$$area\ of\ \Delta GAB = \frac{1}{3}(area\ of\ \Delta ABC) \quad \dots (2)$$

$$area\ of\ \Delta GCA = \frac{1}{2}\left|\overrightarrow{CA} \times \overrightarrow{CG}\right| = \frac{1}{2}\left|\overrightarrow{CA} \times \left(\overrightarrow{OG} - \overrightarrow{OC}\right)\right|$$

$$= \frac{1}{2}\left|\overrightarrow{CA} \times \left(\frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3} - \overrightarrow{OC}\right)\right| = \frac{1}{2}\left|\overrightarrow{CA} \times \left(\frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} - 3\overrightarrow{OC}}{3}\right)\right|$$

$$= \frac{1}{2}\left|\overrightarrow{CA} \times \left(\frac{\overrightarrow{OA} + \overrightarrow{OB} - 2\overrightarrow{OC}}{3}\right)\right| = \frac{1}{2}\left|\overrightarrow{CA} \times \left(\frac{\overrightarrow{OA} - \overrightarrow{OC} + \overrightarrow{OB} - \overrightarrow{OC}}{3}\right)\right|$$

$$= \frac{1}{2}\left|\overrightarrow{CA} \times \left(\frac{\overrightarrow{CA} + \overrightarrow{CB}}{3}\right)\right| = \frac{1}{6}\left|\overrightarrow{CA} \times \left(\overrightarrow{CA} + \overrightarrow{CB}\right)\right|$$

$$= \frac{1}{6}\left|\overrightarrow{CA} \times \overrightarrow{CA} + \overrightarrow{CA} \times \overrightarrow{CB}\right| = \frac{1}{6}\left|\vec{0} + \overrightarrow{CA} \times \overrightarrow{CB}\right| = \frac{1}{6}\left|\overrightarrow{CA} \times \overrightarrow{CB}\right|$$

$$= \frac{1}{3} \times \frac{1}{2}\left|\overrightarrow{CA} \times \overrightarrow{CB}\right|$$

$$area\ of\ \Delta GCA = \frac{1}{3}(area\ of\ \Delta ABC) \quad \dots (3)$$

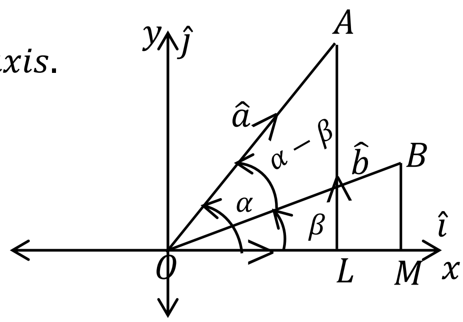
From (1), (2) *and* (3)

$$(area\ of \Delta GAB) = (area\ of \Delta GBC) = (area\ of\ GCA\Delta) = \frac{1}{3}\ (area\ of \Delta\ ABC).$$

9. Prove that $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

Let $\overrightarrow{OA} = \hat{a}, \overrightarrow{OB} = \hat{b}$, *be the unit vectors making angles* α *and* β *with positive* $x-axis$

Draw AL and BM perpendicular to the $x-axis$.



In Right angled ΔOLA

$$cos\alpha = \frac{adj}{hyp} \Rightarrow cos\alpha = \frac{\left|\overrightarrow{OL}\right|}{\left|\overrightarrow{OA}\right|}$$

$$cos\alpha = \frac{\left|\overrightarrow{OL}\right|}{|\hat{a}|} \Rightarrow cos\alpha = \frac{\left|\overrightarrow{OL}\right|}{1} \Rightarrow \boxed{OL = cos\alpha}$$

$$sin\alpha = \frac{opp}{hyp} \Rightarrow sin\alpha = \frac{\left|\overrightarrow{AL}\right|}{\left|\overrightarrow{OA}\right|} \Rightarrow sin\alpha = \frac{\left|\overrightarrow{AL}\right|}{|\hat{a}|} \Rightarrow sin\alpha = \frac{\left|\overrightarrow{AL}\right|}{1}$$

$$\boxed{AL = sin\alpha}$$

$$\overrightarrow{OA} = \overrightarrow{OL} + \overrightarrow{LA} \Rightarrow \overrightarrow{OA} = OL\hat{\imath} + LA\hat{\jmath}$$

$$\boxed{\hat{a} = cos\alpha\hat{\imath} + sin\alpha\hat{\jmath}}$$

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$Similarly, \hat{b} = cos\beta\hat{\imath} + sin\beta\hat{\jmath}$

$The\ angle\ between\ \hat{a}\ and\ \hat{b}\ is\ \alpha - \beta$

$$\hat{a}\,.\hat{b} = |\hat{a}||\hat{b}|\cos(\alpha-\beta)$$

$$\hat{a}\,.\hat{b} = (1)(1)\cos(\alpha-\beta) \Longrightarrow \hat{a}\,.\hat{b} = \cos(\alpha-\beta)\ \dots(1)$$

$$\hat{a}\,.\hat{b} = (cos\alpha\hat{\imath} + sin\alpha\hat{\jmath}).(cos\beta\hat{\imath} + sin\beta\hat{\jmath})$$

$$\hat{a}\,.\hat{b} = cos\alpha\ cos\beta + sin\alpha\ sin\beta\ \dots(2)$$

From (1) and (2)

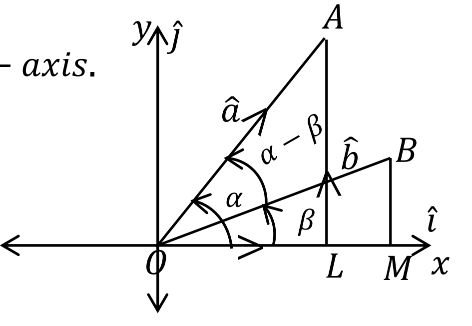
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$\vec{a}\,.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ → angle between $\vec{a}$ and $\vec{b}$

Example : 6.5 Prove that $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Let $\overrightarrow{OA} = \hat{a}, \overrightarrow{OB} = \hat{b}$, be the unit vectors making angles α and β with positive x − axis

Draw AL and BM perpendicular to the x − axis.



In Right angled ΔOLA

$$cos\alpha = \frac{adj}{hyp} \Longrightarrow cos\alpha = \frac{|\overrightarrow{OL}|}{|\overrightarrow{OA}|}$$

$$cos\alpha = \frac{|\overrightarrow{OL}|}{|\hat{a}|} \Longrightarrow cos\alpha = \frac{|\overrightarrow{OL}|}{1} \Longrightarrow OL = cos\alpha$$

$$sin\alpha = \frac{opp}{hyp} \Longrightarrow sin\alpha = \frac{|\overrightarrow{AL}|}{|\overrightarrow{OA}|} \Longrightarrow sin\alpha = \frac{|\overrightarrow{AL}|}{|\hat{a}|} \Longrightarrow sin\alpha = \frac{|\overrightarrow{AL}|}{1}$$

$$\boxed{AL = sin\alpha}$$

$$\overrightarrow{OA} = \overrightarrow{OL} + \overrightarrow{LA} \Longrightarrow \overrightarrow{OA} = OL\hat{\imath} + LA\hat{\jmath}$$

$$\hat{a} = cos\alpha\hat{\imath} + sin\alpha\hat{\jmath}$$

$Similarly, \hat{b} = cos\beta\hat{\imath} + sin\beta\hat{\jmath}$

$The\ angle\ between\ \hat{a}\ and\ \hat{b}\ is\ \alpha - \beta$

$$\hat{b} \times \hat{a} = |\hat{b}||\hat{a}|\sin(\alpha-\beta)\hat{k}$$

$$\hat{b} \times \hat{a} = (1)(1)\sin(\alpha-\beta)\hat{k} \Longrightarrow \hat{b} \times \hat{a} = \sin(\alpha-\beta)\hat{k}\ \dots(1)$$

$$\hat{a} = cos\alpha\hat{\imath} + sin\alpha\hat{\jmath}, \hat{b} = cos\beta\hat{\imath} + sin\beta\hat{\jmath}$$

$$\hat{b} \times \hat{a} = \begin{vmatrix} \overset{+}{\hat{\imath}} & \overset{-}{\hat{\jmath}} & \overset{+}{\hat{k}} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$$= \hat{\imath}(0-0) - \hat{\jmath}(0-0) + \hat{k}\left[\sin\alpha\cos\beta - \cos\alpha\sin\beta\right]$$

$$\hat{b} \times \hat{a} = (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\hat{k}\ \dots(2)$$

From (1) and (2)

$\sin(\alpha-\beta)\hat{k} = (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\hat{k}$

$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$ (θ: angle between $\vec{a}$ and $\vec{b}$)

$$\boldsymbol{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

Example: 6.3 Prove that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Let $\overrightarrow{OA} = \hat{a}, \overrightarrow{OB} = \hat{b}$, be the unit vectors making angles α and β with positive $x-$axis

Draw AL and BM perpendicular to the $x-$axis.

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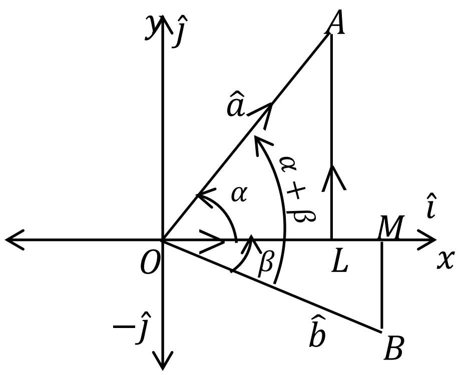

$$\cos\alpha = \frac{adj}{hyp} \Rightarrow \cos\alpha = \frac{|\overrightarrow{OL}|}{|\overrightarrow{OA}|}$$

$$\cos\alpha = \frac{|\overrightarrow{OL}|}{|\hat{a}|} \Rightarrow \cos\alpha = \frac{|\overrightarrow{OL}|}{1} \Rightarrow OL = \cos\alpha$$

$$\sin\alpha = \frac{opp}{hyp} \Rightarrow \sin\alpha = \frac{|\overrightarrow{AL}|}{|\overrightarrow{OA}|} \Rightarrow \sin\alpha = \frac{|\overrightarrow{AL}|}{|\hat{a}|}$$

$$\sin\alpha = \frac{|\overrightarrow{AL}|}{1} \Rightarrow \boxed{AL = \sin\alpha}$$

$$\overrightarrow{OA} = \overrightarrow{OL} + \overrightarrow{LA} \Rightarrow \overrightarrow{OA} = OL\hat{i} + LA\hat{j}$$

$$\hat{a} = \cos\alpha\hat{i} + \sin\alpha\hat{j}$$

Similarly, $\hat{b} = \cos\beta\hat{i} - \sin\beta\hat{j}$

The angle between $\hat{a}$ and $\hat{b}$ is $\alpha+\beta$

$$\hat{a}\,.\,\hat{b} = |\hat{a}||\hat{b}|\cos(\alpha+\beta)$$

$$\hat{a}\,.\,\hat{b} = (1)(1)\cos(\alpha+\beta) \Rightarrow \hat{a}\,.\,\hat{b} = \cos(\alpha+\beta) \;\ldots(1)$$

$$\hat{a}\,.\,\hat{b} = (\cos\alpha\hat{i} + \sin\alpha\hat{j}).(\cos\beta\hat{i} - \sin\beta\hat{j})$$

$$\hat{a}\,.\,\hat{b} = \cos\alpha\cos\beta - \sin\alpha\sin\beta \;\ldots(2)$$

From (1) and (2)

$$\boldsymbol{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

10. Prove that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Let $\overrightarrow{OA} = \hat{a}, \overrightarrow{OB} = \hat{b}$, be the unit vectors making angles α and β with positive $x-$axis

Draw AL and BM perpendicular to the $x-$axis.

In Right angled ΔOLA

$$\cos\alpha = \frac{adj}{hyp} \Rightarrow \cos\alpha = \frac{|\overrightarrow{OL}|}{|\overrightarrow{OA}|}$$

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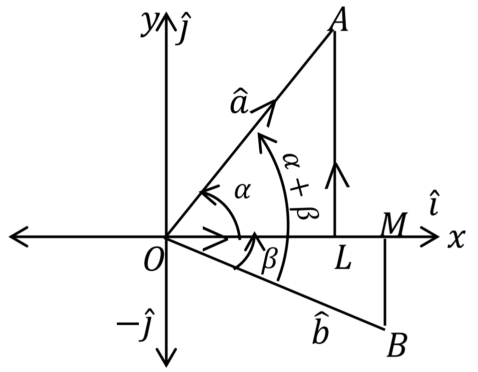
$$\cos\alpha = \frac{|\overrightarrow{OL}|}{|\hat{a}|} \implies \cos\alpha = \frac{|\overrightarrow{OL}|}{1} \implies OL = \cos\alpha$$

$$\sin\alpha = \frac{opp}{hyp} \implies \sin\alpha = \frac{|\overrightarrow{AL}|}{|\overrightarrow{OA}|} \implies \sin\alpha = \frac{|\overrightarrow{AL}|}{|\hat{a}|} \implies \sin\alpha = \frac{|\overrightarrow{AL}|}{1}$$

$$AL = \sin\alpha$$



$$\overrightarrow{OA} = \overrightarrow{OL} + \overrightarrow{LA} \implies \overrightarrow{OA} = OL\hat{\imath} + LA\hat{\jmath}$$

$$\hat{a} = \cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$$

$$Similarly, \hat{b} = \cos\beta\hat{\imath} - \sin\beta\hat{\jmath}$$

The angle between $\hat{a}$ *and* $\hat{b}$ *is* $\alpha + \beta$

$$\hat{b} \times \hat{a} = |\hat{b}||\hat{a}|\sin(\alpha+\beta)\hat{k}$$

$$\hat{b} \times \hat{a} = (1)(1)\sin(\alpha+\beta)\hat{k} \implies \hat{b} \times \hat{a} = \sin(\alpha+\beta)\hat{k} \ldots (1)$$

$$\hat{a} = \cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}, \hat{b} = \cos\beta\hat{\imath} - \sin\beta\hat{\jmath}$$

$$\hat{b} \times \hat{a} = \begin{vmatrix} \overset{+}{\hat{\imath}} & \overset{-}{\hat{\jmath}} & \overset{+}{\hat{k}} \\ \cos\beta & -\sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$$= \hat{\imath}(0-0) - \hat{\jmath}(0-0) + \hat{k}\left[\sin\alpha\cos\beta - \cos\alpha(-\sin\beta)\right]$$

$$\hat{b} \times \hat{a} = (\sin\alpha\cos\beta + \cos\alpha\sin\beta)\hat{k} \ \ldots (2)$$

From (1) *and* (2)

$$\sin(\alpha+\beta)\hat{k} = (\sin\alpha\cos\beta + \cos\alpha\sin\beta)\hat{k}$$

$$\boldsymbol{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

11. ***A particle acted on by constant forces*** $\boldsymbol{8\hat{\imath} + 2\hat{\jmath} - 6\hat{k}}$ ***and*** $\boldsymbol{6\hat{\imath} + 2\hat{\jmath} - 2\hat{k}}$ ***is displaced from the point*** $\boldsymbol{(1,2,3)}$ ***to the point*** $\boldsymbol{(5,4,1)}$***. Find the total work done by the forces.***

Let $\overrightarrow{F_1} = 8\hat{\imath} + 2\hat{\jmath} - 6\hat{k}, \overrightarrow{F_2} = 6\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

$$\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2}$$

$$\vec{F} = 8\hat{\imath} + 2\hat{\jmath} - 6\hat{k} + 6\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$$

$$\vec{F} = 14\hat{\imath} + 4\hat{\jmath} - 8\hat{k}$$

Let $\overrightarrow{OA} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}, \overrightarrow{OB} = 5\hat{\imath} + 4\hat{\jmath} + \hat{k}$

$$\vec{d} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 5\hat{\imath} + 4\hat{\jmath} + \hat{k} - \hat{\imath} - 2\hat{\jmath} - 3\hat{k}$$

$$\vec{d} = 4\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$$

Work done $= \vec{F}.\vec{d}$

$$= (14\hat{\imath} + 4\hat{\jmath} - 8\hat{k}).(4\hat{\imath} + 2\hat{\jmath} - 2\hat{k}) = 56 + 8 + 16$$

Work done = 80 *units*

$$\overrightarrow{F_2} = 10\sqrt{2}\left(\frac{10\hat{\imath}+6\hat{\jmath}-8\hat{k}}{|10\hat{\imath}+6\hat{\jmath}-8\hat{k}|}\right) = 10\sqrt{2}\left(\frac{10\hat{\imath}+6\hat{\jmath}-8\hat{k}}{\sqrt{10^2+6^2+(-8)^2}}\right)$$

$$= 10\sqrt{2}\left(\frac{10\hat{\imath}+6\hat{\jmath}-8\hat{k}}{\sqrt{100+36+64}}\right) = 10\sqrt{2}\left(\frac{10\hat{\imath}+6\hat{\jmath}-8\hat{k}}{\sqrt{200}}\right)$$

$$= 10\sqrt{2}\left(\frac{10\hat{\imath}+6\hat{\jmath}-8\hat{k}}{\sqrt{10\times 10\times 2}}\right) = 10\sqrt{2}\left(\frac{10\hat{\imath}+6\hat{\jmath}-8\hat{k}}{10\sqrt{2}}\right)$$

$$\overrightarrow{F_2} = 10\hat{\imath}+6\hat{\jmath}-8\hat{k}$$

$$\boxed{\vec{F} = \overrightarrow{F_1}+\overrightarrow{F_2}}$$

$$\vec{F} = 3\hat{\imath}+4\hat{\jmath}+5\hat{k}+10\hat{\imath}+6\hat{\jmath}-8\hat{k}$$

$$\vec{F} = 13\hat{\imath}+10\hat{\jmath}-3\hat{k}$$

$$\text{Work done} = \vec{F}.\vec{d}$$

$$= (13\hat{\imath}+10\hat{\jmath}-3\hat{k}).(2\vec{i}+4\vec{j}-\vec{k}) = 26+40+3$$

$$\boxed{\text{Work done} = 69\ units}$$

13. ***Find the magnitude and direction cosines of the torque of a force represented by $3\hat{\imath}+4\hat{\jmath}-5\hat{k}$ about the point with position vector $2\hat{\imath}-3\hat{\jmath}+4\hat{k}$ acting through a point whose position vector $4\hat{\imath}+2\hat{\jmath}-3\hat{k}$.***

$\vec{F} = 3\hat{\imath}+4\hat{\jmath}-5\hat{k}$, $\overrightarrow{OA} = 2\hat{\imath}-3\hat{\jmath}+4\hat{k}$, $\overrightarrow{OB} = 4\hat{\imath}+2\hat{\jmath}-3\hat{k}$

$$\vec{r} = \overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = 4\hat{\imath}+2\hat{\jmath}-3\hat{k}-2\hat{\imath}+3\hat{\jmath}-4\hat{k}$$

$$\vec{r} = 2\hat{\imath}+5\hat{\jmath}-7\hat{k}$$

$$\text{Torque} = \vec{r}\times\vec{F}$$

$$\vec{r}\times\vec{F} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 5 & -7\\ 3 & 4 & -5\end{vmatrix} = \hat{\imath}(-25+28)-\hat{\jmath}(-10+21)+\hat{k}(8-15)$$

$$\vec{r}\times\vec{F} = 3\hat{\imath}-11\hat{\jmath}-7\hat{k}$$

$$\text{Magnitude of the Torque} = |\vec{r}\times\vec{F}|$$

$$= \sqrt{3^2+(-11)^2+(-7)^2} = \sqrt{9+121+49} = \sqrt{179}$$

$$\text{Direction cosines} = \left(\frac{3}{\sqrt{179}}, \frac{-11}{\sqrt{179}}, \frac{-7}{\sqrt{179}}\right)$$

14. ***Find the torque of the resultant of the three forces represented by $-3\hat{\imath}+6\hat{\jmath}-3\hat{k}$, $4\hat{\imath}-10\hat{\jmath}+12\hat{k}$ and $4\hat{\imath}+7\hat{\jmath}$ acting at the point with position vector $8\hat{\imath}-6\hat{\jmath}-4\hat{k}$, about the point with position vector $18\hat{\imath}+3\hat{\jmath}-9\hat{k}$.***

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Let $\overrightarrow{F_1} = -3\hat{\imath} + 6\hat{\jmath} - 3\hat{k}, \overrightarrow{F_2} = 4\hat{\imath} - 10\hat{\jmath} + 12\hat{k}, \overrightarrow{F_3} = 4\hat{\imath} + 7\hat{\jmath}$

$$\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}$$

$$\vec{F} = -3\hat{\imath} + 6\hat{\jmath} - 3\hat{k} + 4\hat{\imath} - 10\hat{\jmath} + 12\hat{k} + 4\hat{\imath} + 7\hat{\jmath}$$

$$\vec{F} = 5\hat{\imath} + 3\hat{\jmath} + 9\hat{k}$$

Let $\overrightarrow{OA} = 18\hat{\imath} + 3\hat{\jmath} - 9\hat{k}, \overrightarrow{OB} = 8\hat{\imath} - 6\hat{\jmath} - 4\hat{k}$

$$\vec{r} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= 8\hat{\imath} - 6\hat{\jmath} - 4\hat{k} - 18\hat{\imath} - 3\hat{\jmath} + 9\hat{k}$$

$$\vec{r} = -10\hat{\imath} - 9\hat{\jmath} + 5\hat{k}$$

$$\boxed{\text{Torque} = \vec{r} \times \vec{F}}$$

$$\vec{r} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -10 & -9 & 5 \\ 5 & 3 & 9 \end{vmatrix}$$

$$= \hat{\imath}(-18 - 15) - \hat{\jmath}(-90 - 25) + \hat{k}(-30 + 45)$$

$$\boxed{\vec{r} \times \vec{F} = -96\hat{\imath} + 115\hat{\jmath} + 15\hat{k}}$$

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EXERCISE : 6.2

Example 6.12: ***If*** $\vec{a} = -3\hat{i} - \hat{j} + 5\hat{k}, \vec{b} = \hat{i} - 2\hat{j} + \hat{k}, \vec{c} = 4\hat{j} - 5\hat{k}$, ***find*** $\vec{a}.(\vec{b} \times \vec{c})$.

$$\vec{a}.(\vec{b} \times \vec{c}) = [\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} -3 & -1 & 5 \\ 1 & -2 & 1 \\ 0 & 4 & -5 \end{vmatrix}$$

$$= -3(10-4) + 1(-5+0) + 5\,(4+0)$$

$$= -3(6) + 1(-5) + 5\,(4) = -18 - 5 + 20$$

$$= -3$$

Example 6.13: ***Find the volume of the parallelepiped whose coterminus edges are given by the vectors*** $-2\hat{i} - 3\hat{j} + 4\hat{k}, \hat{i} + 2\hat{j} - \hat{k}$ ***and*** $3\hat{i} - \hat{j} + 2\hat{k}$.

Let $\vec{a} = -2\hat{i} - 3\hat{j} + 4\hat{k}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}, \vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$

Volume of the parallelopiped $= [\vec{a}\;\vec{b}\;\vec{c}]$

$$[\vec{a}\;\vec{b}\;\vec{c}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 2(4-1) + 3(2+3) + 4\,(-1-6)$$

$$= 2(3) + 3(5) + 4\,(-7) = 6 + 15 - 28$$

$$= -7 = |-7| = 7 \text{ cub. units}$$

Example 6.14: ***Show that the vectors*** $\hat{i} + 2\hat{j} - 3\hat{k}, 2\hat{i} - \hat{j} + 2\hat{k}$ ***and*** $3\hat{i} + \hat{j} - \hat{k}$ ***are coplanar.***

Let $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}, \vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}, \vec{c} = 3\hat{i} + \hat{j} - \hat{k}$

$\vec{a}, \vec{b}, \vec{c}$ *are coplanar if and only if* $[\vec{a}\;\vec{b}\;\vec{c}] = 0$

$$[\vec{a}\;\vec{b}\;\vec{c}] = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix} = 1(1-2) - 2(-2-6) - 3\,(2+3)$$

$$= 1(-1) - 2(-8) - 3\,(5)$$

$$= -1 + 16 - 15 = 0$$

$\therefore$ *The three given vectors are coplanar.*

Example 6.15: ***If*** $2\hat{i} - \hat{j} + 3\hat{k}, 3\hat{i} + 2\hat{j} + \hat{k}, \hat{i} + m\hat{j} + 4\hat{k}$ ***are coplanar, find the value of*** m.

Let $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}, \vec{b} = 3\hat{i} + 2\hat{j} + \hat{k}, \vec{c} = \hat{i} + m\hat{j} + 4\hat{k}$

$$[\vec{a}\;\vec{b}\;\vec{c}] = 0 \Rightarrow \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & m & 4 \end{vmatrix} = 0$$

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$$2(8-m)+1(12-1)+3\,(3m-2)=0$$

$$16-2m+12-1+9m-6=0$$

$$7m+28-7=0 \Rightarrow 7m+21=0$$

$$7m=-21 \Rightarrow m=-\frac{21}{7}$$

$$\boxed{m=-3}$$

Example 6.16**: Show that the points* $(6,-7,0),(16,-19,-4),(0,3,-6)$ $(2,-5,10)$ ***lie on a same plane.

Let $\overrightarrow{OA}=6\hat{\imath}-7\hat{\jmath},\ \overrightarrow{OB}=16\hat{\imath}-19\hat{\jmath}-4\hat{k},\ \overrightarrow{OC}=3\hat{\jmath}-6\hat{k}$ *and*

$$\overrightarrow{OD}=2\hat{\imath}-5\hat{\jmath}+10\hat{k}$$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=16\hat{\imath}-19\hat{\jmath}-4\hat{k}-6\hat{\imath}+7\hat{\jmath}$$

$$\overrightarrow{AB}=10\hat{\imath}-12\hat{\jmath}-4\hat{k}$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=3\hat{\jmath}-6\hat{k}-6\hat{\imath}+7\hat{\jmath}$$

$$\overrightarrow{AC}=-6\hat{\imath}+10\hat{\jmath}-6\hat{k}$$

$$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=2\hat{\imath}-5\hat{\jmath}+10\hat{k}-6\hat{\imath}+7\hat{\jmath}$$

$$\overrightarrow{AD}=-4\hat{\imath}+2\hat{\jmath}+10\hat{k}$$

$$[\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}]=\begin{vmatrix}10 & -12 & -4\\ -6 & 10 & -6\\ -4 & 2 & 10\end{vmatrix}$$

$$=10(100+12)+12(-60-24)-4(-12+40)$$

$$=10(112)+12(-84)-4(28)$$

$$=1120-1008-112=0$$

Hence the above points are lying on the same plane.

***Example 6.17**: If the vectors* $\vec{a},\vec{b},\vec{c}$ *are coplanar, then prove that the vectors* $\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}$ *are also coplanar.*

If $\vec{a},\vec{b},\vec{c}$ are coplanar than $[\vec{a},\vec{b},\vec{c}]=0$ .To prove $\vec{a}+\vec{b}\,,\vec{b}+\vec{c}\,,\vec{c}+\vec{a}$ are coplanar i.e

$$[\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}]=0$$

$$L.H.S.=[\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}]$$

$$=(\vec{a}+\vec{b}).\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}$$

$$=(\vec{a}+\vec{b}).\ \left\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\right\}$$

$$=(\vec{a}+\vec{b}).\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{0}+\vec{c}\times\vec{a}\}$$

$= \vec{a}.(\vec{b}\times\vec{c}) + \vec{a}.(\vec{b}\times\vec{a}) + \vec{a}.(\vec{c}\times\vec{a}) + \vec{b}.(\vec{b}\times\vec{c}) + \vec{b}.(\vec{b}\times\vec{a}) + \vec{b}.(\vec{c}\times\vec{a})$

$= [\vec{a}\,\vec{b}\,\vec{c}] + [\vec{a}\,\vec{b}\,\vec{a}] + [\vec{a}\,\vec{c}\,\vec{a}] + [\vec{b}\,\vec{b}\,\vec{c}] + [\vec{b}\,\vec{b}\,\vec{a}] + [\vec{b}\,\vec{c}\,\vec{a}]$

$= [\vec{a},\vec{b},\vec{c}] + 0 + 0 + 0 + 0 + [\vec{a},\vec{b},\vec{c}]$

$= 2[\vec{a},\vec{b},\vec{c}] = 2(0) = 0$

***Example 6.18**: If $\vec{a},\vec{b},\vec{c}$ are three vectors prove that* $[\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}] = -[\vec{a},\vec{b},\vec{c}]$.

$$[\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}] = \begin{vmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 1 & 1 & 1\end{vmatrix}[\vec{a},\vec{b},\vec{c}]$$

$$=1(1+0)+0+1\,(1-1)\,[\vec{a},\vec{b},\vec{c}]$$

$$= -[\vec{a},\vec{b},\vec{c}]$$

1. *If $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}, \vec{b} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}, \vec{c} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$, find $\vec{a}.(\vec{b}\times\vec{c})$.*

$$\vec{a}.(\vec{b}\times\vec{c}) = [\vec{a},\vec{b},\vec{c}]$$

$$= \begin{vmatrix}1 & -2 & 3\\ 2 & 1 & -2\\ 3 & 2 & 1\end{vmatrix}$$

$$= 1(1+4) + 2(2+6) + 3(4-3)$$

$$= 1(5) + 2(8) + 3(1) = 5 + 16 + 3$$

$$= 24$$

2. *Find the volume of the paralellepiped whose coterminous edges are represented by the vectors $-6\hat{\imath} + 14\hat{\jmath} + 10\hat{k}, 14\hat{\imath} - 10\hat{\jmath} - 6\hat{k}$ and $2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$.*

Let $\vec{a} = -6\hat{\imath} + 14\hat{\jmath} + 10\hat{k}, \vec{b} = 14\hat{\imath} - 10\hat{\jmath} - 6\hat{k}, \vec{c} = 2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$

Volume of the parallelopiped having $\vec{a},\vec{b}$ and $\vec{c}$ as its co − terminus edges is $\vec{a}.(\vec{b}\times\vec{c}) = [\vec{a},\vec{b},\vec{c}]$

$$[\vec{a},\vec{b},\vec{c}] = \begin{vmatrix}-6 & 14 & 10\\ 14 & -10 & -6\\ 2 & 4 & -2\end{vmatrix}$$

$$= -6(20+24) - 14(-28+12) + 10\,(56+20)$$

$$= -6(44) - 14(-16) + 10\,(76)$$

$$= -264 + 224 + 760$$

$$= 720 = 720 \text{ cu.units}$$

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3. *The volume of a parallelepiped whose edges are represented by $7\hat{i}+\lambda\hat{j}-3\hat{k}, \hat{i}+2\hat{j}-\hat{k}, -3\hat{i}+7\hat{j}+5\hat{k}$ is 90 cubic units. Find the value of λ.*

$$Let\ \vec{a}=7\hat{i}+\lambda\hat{j}-3\hat{k}, \vec{b}=\hat{i}+2\hat{j}-\hat{k}, \vec{c}=-3\hat{i}+7\hat{j}+5\hat{k}$$

$Volume\ of\ the\ parallelopiped = 90$

$$[\vec{a},\vec{b},\vec{c}]=90$$

$$\begin{vmatrix} 7 & \lambda & -3 \\ 1 & 2 & -1 \\ -3 & 7 & 5 \end{vmatrix} = 90 \Rightarrow 7(10+7)-\lambda(5-3)-3(7+6)=90$$

$$7(17)-2\lambda-3(13)=90 \Rightarrow 119-2\lambda-39=90$$

$$80-2\lambda=90 \Rightarrow -2\lambda=90-80$$

$$-2\lambda=10 \Rightarrow \lambda=\frac{10}{-2}$$

$$\boxed{\lambda=-5}$$

4. *If $\vec{a},\vec{b},\vec{c}$ are three non − coplanar vectors represented by con − current edges of a parallelepiped of volume 4 cubic units, find the value of $(\vec{a}+\vec{b}).(\vec{b}\times\vec{c})+(\vec{b}+\vec{c}).(\vec{c}\times\vec{a})+(\vec{c}+\vec{a}).(\vec{a}\times\vec{b})$.*

$$[\vec{a},\vec{b},\vec{c}]=\pm 4$$

$$(\vec{a}+\vec{b}).(\vec{b}\times\vec{c})+(\vec{b}+\vec{c}).(\vec{c}\times\vec{a})+(\vec{c}+\vec{a}).(\vec{a}\times\vec{b})$$

$$=\vec{a}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{c}\times\vec{a})+\vec{c}.(\vec{c}\times\vec{a})+\vec{c}.(\vec{a}\times\vec{b})+\vec{a}.(\vec{a}\times\vec{b})$$

$$=[\vec{a},\vec{b},\vec{c}]+[\vec{b},\vec{b},\vec{c}]+[\vec{b},\vec{c},\vec{a}]+[\vec{c},\vec{c},\vec{a}]+[\vec{c},\vec{a},\vec{b}]+[\vec{a},\vec{a},\vec{b}]$$

$$=[\vec{a},\vec{b},\vec{c}]+0+[\vec{b},\vec{c},\vec{a}]+0+[\vec{c},\vec{a},\vec{b}]+0$$

$$=[\vec{a},\vec{b},\vec{c}]+[\vec{a},\vec{b},\vec{c}]+[\vec{a},\vec{b},\vec{c}]$$

$$=3[\vec{a},\vec{b},\vec{c}] \qquad \left[\because [\vec{a},\vec{b},\vec{c}]=[\vec{b},\vec{c},\vec{a}]=[\vec{c},\vec{a},\vec{b}]\right]$$

$$=3(\pm 4)=\pm 12$$

5. *Find the altitude of a parallelepiped determined by the vectors $\vec{a}=-2\hat{i}+5\hat{j}+3\hat{k}, \vec{b}=\hat{i}+3\hat{j}-2\hat{k}$ and $\vec{c}=-3\hat{i}+\hat{j}+4\hat{k}$ if the base is taken as the parallelogram determined by $\vec{b}$ and $\vec{c}$.*

$$Given:\ \vec{a}=-2\hat{i}+5\hat{j}+3\hat{k}, \vec{b}=\hat{i}+3\hat{j}-2\hat{k}, \vec{c}=-3\hat{i}+\hat{j}+4\hat{k}$$

$$Volume\ of\ the\ parallelepiped=\vec{a}.(\vec{b}\times\vec{c})=[\vec{a},\vec{b},\vec{c}]$$

$$=\begin{vmatrix} -2 & 5 & 3 \\ 1 & 3 & -2 \\ -3 & 1 & 4 \end{vmatrix} = -2(12+2)-5(4-6)+3(1+9)$$

$$= -2(14) - 5(-2) + 3(10) = -28 + 10 + 30$$

Volume of the parallelepiped $= 12$

Area of the base parallelogram $= |\vec{b} \times \vec{c}|$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -2 \\ -3 & 1 & 4 \end{vmatrix} = \hat{\imath}(12+2) - \hat{\jmath}(4-6) + \hat{k}(1+9)$$

$$\vec{b} \times \vec{c} = 14\hat{\imath} + 2\hat{\jmath} + 10\hat{k}$$

Area of the parallelogram $= |\vec{b} \times \vec{c}| = \sqrt{14^2 + 2^2 + 10^2} = \sqrt{196 + 4 + 100}$

$$= \sqrt{300} = \sqrt{3 \times 100}$$

Area of the parallelogram $= 10\sqrt{3}$

Volume of the parallelepiped $=$ *Base Area* $\times$ *altitude*

$$\therefore Altitude = \frac{Volume}{Base\ area}$$

$$= \frac{12}{10\sqrt{3}} = \frac{6}{5\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{5 \times 3} = \frac{2\sqrt{3}}{5} \text{ units}$$

6. Determine whether the three vectors $2\hat{\imath} + 3\hat{\jmath} + \hat{k}, \hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ and $3\hat{\imath} + \hat{\jmath} + 3\hat{k}$ are coplanar.

Let $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}, \vec{b} = \hat{\imath} - 2\hat{\jmath} + 2\hat{k}, \vec{c} = 3\hat{\imath} + \hat{\jmath} + 3\hat{k}$

$\vec{a}, \vec{b}$ *and* $\vec{c}$ *are coplanar if* $[\vec{a}, \vec{b}, \vec{c}] = 0$

$$= \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & 1 & 3 \end{vmatrix} = 2(-6-2) - 3(3-6) + 1(1+6)$$

$$= 2(-8) - 3(-3) + 1(7) = -16 + 9 + 7 = 0$$

Hence, the givne vectors are co $-$ *planar.*

7. Let $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}, \vec{b} = \hat{\imath}$ and $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$. If $c_1 = 1$ and $c_2 = 2$, find c_3 such that $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

Given: $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}, \vec{b} = \hat{\imath}, \vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$

$c_1 = 1$ *and* $c_2 = 2$

$$\therefore \vec{c} = \hat{\imath} + 2\hat{\jmath} + c_3\hat{k}$$

$\vec{a}, \vec{b}$ *and* $\vec{c}$ *are coplanar.* $\therefore [\vec{a}, \vec{b}, \vec{c}] = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & c_3 \end{vmatrix} = 0 \Longrightarrow 1(0-0) - 1(c_3 - 0) + 1(2-0) = 0$$

$$0 - c_3 + 2 = 0 \implies -c_3 = -2$$

$$\boxed{c_3 = 2}$$

8. *If $\vec{a} = \hat{\imath} - \hat{k}, \vec{b} = x\hat{\imath} + \hat{\jmath} + (1-x)\hat{k}, \vec{c} = y\hat{\imath} + x\hat{\jmath} + (1+x-y)\hat{k}$, show that $[\vec{a}, \vec{b}, \vec{c}]$ depends on neither x nor y.*

Given: $\vec{a} = \hat{\imath} - \hat{k}, \vec{b} = x\hat{\imath} + \hat{\jmath} + (1-x)\hat{k}, \vec{c} = y\hat{\imath} + x\hat{\jmath} + (1+x-y)\hat{k}$

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix} = 1[(1+x-y) - x(1-x)] - 0 - 1[x^2 - y]$$

$$= 1 + x - y - x + x^2 - x^2 + y = 1$$

$\therefore [\vec{a}, \vec{b}, \vec{c}] = 1$ *for all values of x and y*

$\therefore [\vec{a}, \vec{b}, \vec{c}]$ *depends on neither x nor y.*

9. *If the vectors $a\hat{\imath} + a\hat{\jmath} + c\hat{k}, \hat{\imath} + \hat{k}, c\hat{\imath} + c\hat{\jmath} + b\hat{k}$ are coplanar, prove that c is the geometric mean of a and b.*

Let $\vec{a} = a\hat{\imath} + a\hat{\jmath} + c\hat{k}, \vec{b} = \hat{\imath} + \hat{k}, \vec{c} = c\hat{\imath} + c\hat{\jmath} + b\hat{k}$

Given $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar. $\therefore [\vec{a}, \vec{b}, \vec{c}] = 0$

$$\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0 \implies a(0-c) - a(b-c) + c(c-0) = 0$$

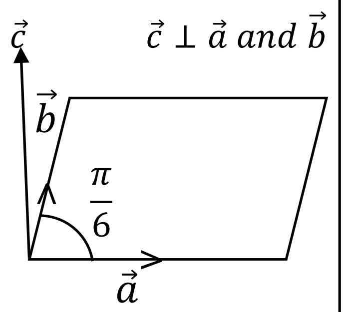
$$-ac - ab + ac + c^2 = 0$$

$$c^2 = ab \implies c = \sqrt{ab}$$

Hence c is the geometric mean of a and b.

10. *Let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, show that $[\vec{a}, \vec{b}, \vec{c}]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$*

Given: $|\vec{c}| = 1$ *and angle between $\vec{a}$ and $\vec{b}$ is* $\frac{\pi}{6}$



$$[\vec{a}, \vec{b}, \vec{c}] = \vec{a}.(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}).\vec{c}$$

$$= \left(|\vec{a}||\vec{b}| \sin\frac{\pi}{6}\vec{c}\right).\vec{c} = |\vec{a}||\vec{b}|\frac{1}{2}(\vec{c}.\vec{c})$$

$$[\vec{a}, \vec{b}, \vec{c}] = |\vec{a}||\vec{b}|\frac{1}{2}$$

$$[\vec{a}, \vec{b}, \vec{c}]^2 = |\vec{a}|^2|\vec{b}|^2\frac{1}{4} \implies [\vec{a}, \vec{b}, \vec{c}]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$$

$[\because \vec{c}.\vec{c} = 1]$

EXERCISE : 6.3

Theorem 6.9: Prove that $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=\vec{0}$

$L.H.S.=\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})$

$=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}+(\vec{b}.\vec{a})\vec{c}-(\vec{b}.\vec{c})\vec{a}+(\vec{c}.\vec{b})\vec{a}-(\vec{c}.\vec{a})\vec{b}$

$=\vec{0}=R.H.S$

Theorem 6.10: Scalar product of four vectors

To prove : $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{a}.\vec{c} & \vec{a}.\vec{d}\\ \vec{b}.\vec{c} & \vec{b}.\vec{d}\end{vmatrix}$

$L.H.S=(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})$

$Let\ \vec{x}=\vec{c}\times\vec{d}$

$=(\vec{a}\times\vec{b}).\vec{x}$ (interchange dot and cross)

$=\vec{a}.(\vec{b}\times\vec{x})=\vec{a}.\{\vec{b}\times(\vec{c}\times\vec{d})\}$

$=\vec{a}.\{(\vec{b}.\vec{d})\vec{c}-(\vec{b}.\vec{c})\vec{d}\}=(\vec{a}.\vec{c})(\vec{b}.\vec{d})-(\vec{b}.\vec{c})(\vec{a}.\vec{d})$

$=\begin{vmatrix}\vec{a}.\vec{c} & \vec{a}.\vec{d}\\ \vec{b}.\vec{c} & \vec{b}.\vec{d}\end{vmatrix}$

Example 6.19: Prove that $[\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}]=[\vec{a},\vec{b},\vec{c}]^2$

L.H.S $=[\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}]$

$=(\vec{a}\times\vec{b}).\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\}$

$=(\vec{a}\times\vec{b}).\{[\vec{b},\vec{c},\vec{a}]\vec{c}-[\vec{b},\vec{c},\vec{c}]\vec{a}\}$

$=(\vec{a}\times\vec{b}).\{[\vec{a},\vec{b},\vec{c}]\vec{c}-\vec{0}\}$

$=(\vec{a}\times\vec{b}).\{[\vec{a},\vec{b},\vec{c}]\vec{c}\}=[\vec{a},\vec{b},\vec{c}](\vec{a}\times\vec{b}).\vec{c}$

$=[\vec{a},\vec{b},\vec{c}][\vec{a},\vec{b},\vec{c}]=[\vec{a},\vec{b},\vec{c}]^2$

$[\vec{b},\vec{c},\vec{c}]=0$

$[\vec{a},\vec{b},\vec{c}]=scalar$

$\vec{a}.(m\vec{b})=m(\vec{a}.\vec{b})$

Example 6.20: Prove that $(\vec{a}.(\vec{b}\times\vec{c}))\vec{a}=(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c})$

$R.H.S=(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c})$

$=[\vec{a}\,\vec{b}\,\vec{c}]\vec{a}-[\vec{a}\,\vec{b}\,\vec{a}]\vec{c}$

$=[\vec{a}\,\vec{b}\,\vec{c}]\vec{a}-(0)\vec{c}=[\vec{a}\,\vec{b}\,\vec{c}]\vec{a}=(\vec{a}.(\vec{b}\times\vec{c}))\vec{a}$

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Example 6.21: For any four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$, we have $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d} = [\vec{a},\vec{c},\vec{d}]\vec{b} - [\vec{b},\vec{c},\vec{d}]\vec{a}$.

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$$

Take $\vec{p} = \vec{a}\times\vec{b}$

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{p}\times(\vec{c}\times\vec{d})$$
$$= (\vec{p}.\vec{d})\vec{c} - (\vec{p}.\vec{c})\vec{d}$$
$$= \left((\vec{a}\times\vec{b}).\vec{d}\right)\vec{c} - \left((\vec{a}\times\vec{b}).\vec{c}\right)\vec{d}$$
$$= [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}$$

Similarly, Take $\vec{q} = \vec{c}\times\vec{d}$

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b})\times\vec{q}$$
$$= (\vec{a}.\vec{q})\vec{b} - (\vec{b}.\vec{q})\vec{a}$$
$$= \left(\vec{a}.(\vec{c}\times\vec{d})\right)\vec{b} - \left(\vec{b}.(\vec{c}\times\vec{d})\right)\vec{a}$$
$$= [\vec{a},\vec{c},\vec{d}]\vec{b} - [\vec{b},\vec{c},\vec{d}]\vec{a}$$

Example 6.22 : If $\vec{a} = -2\hat{\imath}+3\hat{\jmath}-2\hat{k}, \vec{b} = 3\hat{\imath}-\hat{\jmath}+3\hat{k}, \vec{c} = 2\hat{\imath}-5\hat{\jmath}+\hat{k}$, find $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$. State whether they are equal.

To find $(\vec{a}\times\vec{b})\times\vec{c}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 3 & -2 \\ 3 & -1 & 3 \end{vmatrix} = \hat{\imath}(9-2) - \hat{\jmath}(-6+6) + \hat{k}\,(2-9)$$

$$\vec{a}\times\vec{b} = 7\hat{\imath} - 7\hat{k}$$

$$(\vec{a}\times\vec{b})\times\vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 7 & 0 & -7 \\ 2 & -5 & 1 \end{vmatrix} = \hat{\imath}(0-35) - \hat{\jmath}(7+14) + \hat{k}\,(-35-0)$$

$$(\vec{a}\times\vec{b})\times\vec{c} = -35\hat{\imath} - 21\hat{\jmath} - 35\hat{k} \ldots (1)$$

To find $\vec{a}\times(\vec{b}\times\vec{c})$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 3 \\ 2 & -5 & 1 \end{vmatrix} = \hat{\imath}(-1+15) - \hat{\jmath}(3-6) + \hat{k}\,(-15+2)$$

$$\vec{b}\times\vec{c} = 14\hat{\imath} + 3\hat{\jmath} - 13\hat{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ -2 & 3 & -2\\ 14 & 3 & -13\end{vmatrix} = \hat{\imath}(-39+6)-\hat{\jmath}(26+28)+\hat{k}\,(-6-42)$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = -33\hat{\imath}-54\hat{\jmath}-48\hat{k} \quad \ldots(2)$$

From (1) & (2)

$$\therefore\ \vec{a}(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$$

Example 6.23: Let $\vec{a}=\hat{\imath}-\hat{\jmath}, \vec{b}=\hat{\imath}-\hat{\jmath}-4\hat{k}, \vec{c}=3\hat{\jmath}-\hat{k}$ and $\vec{d}=2\hat{\imath}+5\hat{\jmath}+\hat{k}$ be any four vectors then

(i) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$

(ii) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b}-[\vec{b}\,\vec{c}\vec{d}]\vec{a}$

(i) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$

$$\text{L.H.S} = (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$$

$$\vec{a}\times\vec{b} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & -1 & 0\\ 1 & -1 & -4\end{vmatrix} = \hat{\imath}(4-0)-\hat{\jmath}(-4+0)+\hat{k}\,(-1+1)$$

$$\boxed{\vec{a}\times\vec{b} = 4\hat{\imath}+4\hat{\jmath}}$$

$$\vec{c}\times\vec{d} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 0 & 3 & -1\\ 2 & 5 & 1\end{vmatrix} = \hat{\imath}(3+5)-\hat{\jmath}(0+2)+\hat{k}\,(0-6)$$

$$\boxed{\vec{c}\times\vec{d} = 8\hat{\imath}-2\hat{\jmath}-6\hat{k}}$$

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 4 & 4 & 0\\ 8 & -2 & -6\end{vmatrix} = \hat{\imath}(-24+0)-\hat{\jmath}(-24+0)+\hat{k}\,(-8-32)$$

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = -24\hat{\imath}+24\hat{\jmath}-40\hat{k}$$

$$R.H.S = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$$

$$= \begin{vmatrix}1 & -1 & 0\\ 1 & -1 & -4\\ 2 & 5 & 1\end{vmatrix} = 1(-1+20)+1(1+8)-0$$

$$= 1(19)+1(9)$$

$$\boxed{[\vec{a}\,\vec{b}\,\vec{d}] = 28}$$

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$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 0 & 3 & -1 \end{vmatrix} = 1(1+12) + 1(-1+0) - 0$$

$$= 1(13) + 1(-1)$$

$$\boxed{[\vec{a}\,\vec{b}\,\vec{c}] = 12}$$

$$[\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d} = 28(3\hat{j} - \hat{k}) - 12(2\hat{\imath} + 5\hat{j} + \hat{k})$$

$$= 84\hat{j} - 28\hat{k} - 24\hat{\imath} - 60\hat{j} - 12\hat{k}$$

$$[\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d} = -24\hat{\imath} + 24\hat{j} - 40\hat{k} \quad \dots (1)$$

From (1) *and* (2)

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$$

1. $\boldsymbol{If\ \vec{a} = \hat{\imath} - 2\hat{j} + 3\hat{k}, \vec{b} = 2\hat{\imath} + \hat{j} - 2\hat{k}, \vec{c} = 3\hat{\imath} + 2\hat{j} + \hat{k},}$

$\boldsymbol{find\ (i)\ (\vec{a} \times \vec{b}) \times \vec{c}\ (ii)\ \vec{a} \times (\vec{b} \times \vec{c}).}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 1 & -2 \end{vmatrix} = \hat{\imath}(4-3) - \hat{j}(-2-6) + \hat{k}\,(1+4)$$

$$\boxed{\vec{a} \times \vec{b} = \hat{\imath} + 8\hat{j} + 5\hat{k}}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{j} & \hat{k} \\ 1 & 8 & 5 \\ 3 & 2 & 1 \end{vmatrix} = \hat{\imath}(8-10) - \hat{j}(1-15) + \hat{k}\,(2-24)$$

$$\boxed{(\vec{a} \times \vec{b}) \times \vec{c} = -2\hat{\imath} + 14\hat{j} - 22\hat{k}}$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{vmatrix} = \hat{\imath}(1+4) - \hat{j}(2+6) + \hat{k}\,(4-3)$$

$$\boxed{\vec{b} \times \vec{c} = 5\hat{\imath} - 8\hat{j} + \hat{k}}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 5 & -8 & 1 \end{vmatrix} = \hat{\imath}(-2+24) - \hat{j}(1-15) + \hat{k}\,(-8+10)$$

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c}) = 22\hat{\imath} + 14\hat{j} + 2\hat{k}}$$

$\boldsymbol{(ii)(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\vec{d}]\vec{a}}$

$$L.H.S = (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix} = \hat{\imath}(4-0) - \hat{\jmath}(-4+0) + \hat{k}\,(-1+1)$$

$$\boxed{\vec{a}\times\vec{b} = 4\hat{\imath} + 4\hat{\jmath}}$$

$$\vec{c}\times\vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = \hat{\imath}(3+5) - \hat{\jmath}(0+2) + \hat{k}\,(0-6)$$

$$\boxed{\vec{c}\times\vec{d} = 8\hat{\imath} - 2\hat{\jmath} - 6\hat{k}}$$

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix}$$

$$= \hat{\imath}(-24+0) - \hat{\jmath}(-24+0) + \hat{k}\,(-8-32)$$

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = -24\hat{\imath} + 24\hat{\jmath} - 40\hat{k} \quad \ldots(1)$$

$$R.H.S = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

$$[\vec{a}\ \vec{c}\ \vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 1(3+5) + 1(0+2) - 0$$

$$= 1(8) + 1(2)$$

$$\boxed{[\vec{a}\ \vec{c}\ \vec{d}] = 10}$$

$$[\vec{b}\ \vec{c}\ \vec{d}] = \begin{vmatrix} 1 & -1 & -4 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 1(3+5) + 1(0+2) - 4(0-6)$$

$$= 1(8) + 1(2) - 4(-6) = 8 + 2 + 24$$

$$\boxed{[\vec{b}\ \vec{c}\ \vec{d}] = 34}$$

$$[\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} = 10(\hat{\imath} - \hat{\jmath} - 4\hat{k}) - 34(\hat{\imath} - \hat{\jmath})$$

$$= 10\hat{\imath} - 10\hat{\jmath} - 40\hat{k} - 34\hat{\imath} + 34\hat{\jmath}$$

$$[\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} = -24\hat{\imath} + 24\hat{\jmath} - 40\hat{k} \quad \ldots(2)$$

From (1) *and* (2)

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

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2. *For any vector $\vec{a}$. Prove that* $\hat{\imath} \times (\vec{a} \times \hat{\imath}) + \hat{\jmath} \times (\vec{a} \times \hat{\jmath}) + \hat{k} \times (\vec{a} \times \hat{k}) = 2\vec{a}$

$$L.H.S. = \hat{\imath} \times (\vec{a} \times \hat{\imath}) + \hat{\jmath} \times (\vec{a} \times \hat{\jmath}) + \hat{k} \times (\vec{a} \times \hat{k})$$

$$= (\hat{\imath}.\hat{\imath})\vec{a} - (\hat{\imath}.\vec{a})\hat{\imath} + (\hat{\jmath}.\hat{\jmath})\vec{a} - (\hat{\jmath}.\vec{a})\hat{\jmath} + (\hat{k}.\hat{k})\vec{a} - (\hat{k}.\vec{a})\hat{k}$$

$$= (1)\vec{a} - (\hat{\imath}.\vec{a})\hat{\imath} + (1)\vec{a} - (\hat{\jmath}.\vec{a})\hat{\jmath} + (1)\vec{a} - (\hat{k}.\vec{a})\hat{k}$$

$$= \vec{a} - (\hat{\imath}.\vec{a})\hat{\imath} + \vec{a} - (\hat{\jmath}.\vec{a})\hat{\jmath} + \vec{a} - (\hat{k}.\vec{a})\hat{k}$$

$$= 3\vec{a} - (\hat{\imath}.\vec{a})\hat{\imath} - (\hat{\jmath}.\vec{a})\hat{\jmath} - (\hat{k}.\vec{a})\hat{k} = 3\vec{a} - \{(\hat{\imath}.\vec{a})\hat{\imath} + (\hat{\jmath}.\vec{a})\hat{\jmath} + (\hat{k}.\vec{a})\hat{k}\}$$

$$= 3\vec{a} - \vec{a} = 2\vec{a}$$

Let $\vec{a} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$\hat{\imath}.\vec{a} = \hat{\imath}.(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) = x$$

$$\hat{\jmath}.\vec{a} = \hat{\jmath}.(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) = y$$

$$\hat{k}.\vec{a} = \hat{k}.(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) = z$$

$$(\hat{\imath}.\vec{a})\hat{\imath} + (\hat{\jmath}.\vec{a})\hat{\jmath} + (\hat{k}.\vec{a})\hat{k} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$(\hat{\imath}.\vec{a})\hat{\imath} + (\hat{\jmath}.\vec{a})\hat{\jmath} + (\hat{k}.\vec{a})\hat{k} = \vec{a}$$

3. *Prove that* $[\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}] = 0$

$$L.H.S. = [\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}]$$

$$= (\vec{a} - \vec{b}) . [(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})]$$

$$= (\vec{a} - \vec{b}).[\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{c} \times \vec{c} + \vec{c} \times \vec{a}] \qquad [\because \vec{c} \times \vec{c} = \vec{0}]$$

$$= (\vec{a} - \vec{b}).[\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{0} + \vec{c} \times \vec{a}]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}] - [\vec{a}\,\vec{b}\,\vec{a}] + [\vec{a}\,\vec{c}\,\vec{a}] - [\vec{b}\,\vec{b}\,\vec{c}] + [\vec{b}\,\vec{b}\,\vec{a}] - [\vec{b}\,\vec{c}\,\vec{a}]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}] - 0 + 0 - 0 + 0 - [\vec{b}\,\vec{c}\,\vec{a}]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}] - [\vec{a}\,\vec{b}\,\vec{c}] \qquad [\because [\vec{a}\,\vec{b}\,\vec{a}] = [\vec{b}\,\vec{b}\,\vec{c}] = 0]$$

$$= 0 = R.H.S$$

4. If $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}, \vec{b} = 3\hat{\imath} + 5\hat{\jmath} + 2\hat{k}, \vec{c} = -\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$, verify that

(i) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$ (ii) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$.

(i) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

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$$\text{L.H.S} = (\vec{a} \times \vec{b}) \times \vec{c}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ 3 & 5 & 2 \end{vmatrix} = \hat{\imath}(6+5) - \hat{\jmath}(4+3) + \hat{k}\,(10-9)$$

$$\boxed{\vec{a} \times \vec{b} = 11\hat{\imath} - 7\hat{\jmath} + \hat{k}}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 11 & -7 & 1 \\ -1 & -2 & 3 \end{vmatrix} = \hat{\imath}(-21+2) - \hat{\jmath}(33+1) + \hat{k}\,(-22-7)$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = -19\hat{\imath} - 34\hat{\jmath} - 29\hat{k} \quad \ldots (1)$$

$$R.H.S = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$$

$$\vec{a}.\vec{c} = (2\hat{\imath} + 3\hat{\jmath} - \hat{k}).(-\hat{\imath} - 2\hat{\jmath} + 3\hat{k}) = -2 - 6 - 3$$

$$\boxed{\vec{a}.\vec{c} = -11}$$

$$\vec{b}.\vec{c} = (3\hat{\imath} + 5\hat{\jmath} + 2\hat{k}).(-\hat{\imath} - 2\hat{\jmath} + 3\hat{k}) = -3 - 10 + 6$$

$$\boxed{\vec{b}.\vec{c} = -7}$$

$$(\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a} = -11(3\hat{\imath} + 5\hat{\jmath} + 2\hat{k}) + 7(2\hat{\imath} + 3\hat{\jmath} - \hat{k})$$

$$= -33\hat{\imath} - 55\hat{\jmath} - 22\hat{k} + 14\hat{\imath} + 21\hat{\jmath} - 7\hat{k}$$

$$= -19\hat{\imath} - 34\hat{\jmath} - 29\hat{k}$$

$$\boxed{(\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a} = 19\hat{\imath} - 34\hat{\jmath} - 29\hat{k}} \ldots (2)$$

From (1) *and* (2)

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$$

(ii) $\boldsymbol{\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}}$

$$L.H.S = \vec{a} \times (\vec{b} \times \vec{c})$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 5 & 2 \\ -1 & -2 & 3 \end{vmatrix} = \hat{\imath}(15+4) - \hat{\jmath}(9+2) + \hat{k}\,(-6+5)$$

$$\boxed{\vec{b} \times \vec{c} = 19\hat{\imath} - 11\hat{\jmath} - \hat{k}}$$

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$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ 19 & -11 & -1 \end{vmatrix}$$

$$= \hat{\imath}(-3-11) - \hat{\jmath}(-2+19) + \hat{k}\,(-22-57)$$

$$(\vec{a}\times\vec{b})\times\vec{c} = -14\hat{\imath} - 17\hat{\jmath} - 79\hat{k} \quad \dots(1)$$

$$R.H.S = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

$$\vec{a}.\vec{c} = (2\hat{\imath}+3\hat{\jmath}-\hat{k}).(-\hat{\imath}-2\hat{\jmath}+3\hat{k}) = -2-6-3$$

$$\boxed{\vec{a}.\vec{c} = -11}$$

$$\vec{a}.\vec{b} = (2\hat{\imath}+3\hat{\jmath}-\hat{k}).(3\hat{\imath}+5\hat{\jmath}+2\hat{k}) = 6+15-2$$

$$\boxed{\vec{a}.\vec{b} = 19}$$

$$(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = -11(3\hat{\imath}+5\hat{\jmath}+2\hat{k}) - 19(-\hat{\imath}-2\hat{\jmath}+3\hat{k})$$

$$= -33\hat{\imath} - 55\hat{\jmath} - 22\hat{k} + 19\hat{\imath} + 38\hat{\jmath} - 57\hat{k}$$

$$= -14\hat{\imath} - 17\hat{\jmath} - 79\hat{k}$$

$$(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = -14\hat{\imath} - 17\hat{\jmath} - 79\hat{k} \quad \dots(2)$$

From (1) *and* (2) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

5. $\boldsymbol{\vec{a} = 2\hat{\imath}+3\hat{\jmath}-\hat{k}, \vec{b} = -\hat{\imath}+2\hat{\jmath}-4\hat{k}, \vec{c} = \hat{\imath}+\hat{\jmath}+\hat{k}}$, ***then find the value of*** $(\boldsymbol{\vec{a}\times\vec{b}}).(\boldsymbol{\vec{a}\times\vec{c}})$.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ -1 & 2 & -4 \end{vmatrix} = \hat{\imath}(-12+2) - \hat{\jmath}(-8+1) + \hat{k}\,(4+3)$$

$$\boxed{\vec{a}\times\vec{b} = -10\hat{\imath} + 9\hat{\jmath} + 7\hat{k}}$$

$$\vec{a}\times\vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \hat{\imath}(3+1) - \hat{\jmath}(2+1) + \hat{k}\,(2-3)$$

$$\boxed{\vec{a}\times\vec{c} = 4\hat{\imath} - 3\hat{\jmath} - \hat{k}}$$

$$\therefore (\vec{a}\times\vec{b}).(\vec{a}\times\vec{c}) = (-10\hat{\imath}+9\hat{\jmath}+7\hat{k}).(4\hat{\imath}-3\hat{\jmath}-\hat{k})$$

$$= -40-27-7$$

$$\boxed{(\vec{a}\times\vec{b}).(\vec{a}\times\vec{c}) = -74}$$

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6. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar vectors, then show that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{0}$.

If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar vectors then $\vec{a}, \vec{b}, \vec{c}$ are coplanar or $\vec{a}, \vec{b}, \vec{d}$ are also coplanar

Hence $[\vec{a}\,\vec{b}\,\vec{c}] = 0$ and $[\vec{a}\,\vec{b}\,\vec{d}] = 0$

$$L.H.S = (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$$

$$= 0(\vec{c}) - 0(\vec{d}) = \vec{0} = R.H.S$$

7. If $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}, \vec{b} = 2\hat{\imath} - \hat{\jmath} + \hat{k}, \vec{c} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $\vec{a}\times(\vec{b}\times\vec{c}) = l\vec{a} + m\vec{b} + n\vec{c}$, find the values of l, m, n.

Given $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}, \vec{b} = 2\hat{\imath} - \hat{\jmath} + \hat{k}, \vec{c} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

$$\vec{a}.\vec{c} = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}).(3\hat{\imath} + 2\hat{\jmath} + \hat{k}) = 3 + 4 + 3$$

$$\boxed{\vec{a}.\vec{c} = 10}$$

$$\vec{a}.\vec{b} = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}).(2\hat{\imath} - \hat{\jmath} + \hat{k}) = 2 - 2 + 3$$

$$\boxed{\vec{a}.\vec{b} = 3}$$

$$(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = 10\vec{b} - 3\vec{c}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

$$l\vec{a} + m\vec{b} + n\vec{c} = 10\vec{b} - 3\vec{c} \Longrightarrow l\vec{a} + m\vec{b} + n\vec{c} = 0\vec{a} + 10\vec{b} - 3\vec{c}$$

Equating the co − efficients of like terms

$$l = 0,\ m = 10,\ n = -3$$

8. If $\hat{a}, \hat{b}, \hat{c}$ are three unit vectors such that $\hat{b}$ and $\hat{c}$ are non − parallel and $\hat{a}\times(\hat{b}\times\hat{c}) = \frac{1}{2}\hat{b}$, find the angle between $\hat{a}$ and $\hat{c}$

$$\hat{a}\times(\hat{b}\times\hat{c}) = \frac{1}{2}\hat{b} \Longrightarrow (\hat{a}.\hat{c})\hat{b} - (\hat{a}.\hat{b})\hat{c} = \frac{1}{2}\hat{b}$$

here $\hat{a}$ and $\hat{b}$ are perpendicular then $\hat{a}.\hat{b} = 0$ $\quad \boxed{[\because |\hat{a}| = |\hat{c}| = 1]}$

$$(\hat{a}.\hat{c})\hat{b} - (0)\hat{c} = \frac{1}{2}\hat{b} \Longrightarrow (\hat{a}.\hat{c})\hat{b} = \frac{1}{2}\hat{b} \Longrightarrow \hat{a}.\hat{c} = \frac{1}{2} \Longrightarrow |\hat{a}||\hat{c}|\cos\theta = \frac{1}{2}$$

$$(1)(1)\cos\theta = \frac{1}{2} \Longrightarrow \cos\theta = \frac{1}{2}$$

$$\theta = 60^\circ \Longrightarrow \boxed{\theta = \frac{\pi}{3}}$$

EXERCISE : 6.4

Example 6.24: A straight line passes through the point $(1,2,-3)$ and parallel to $4\hat{i}+5\hat{j}-7\hat{k}$. Find (i) vector equation in parametric form (ii) vector equation in non − parametric form (iii) Cartesian equations of the straight line.

Let $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=4\hat{i}+5\hat{j}-7\hat{k}$

(i) vector equation in parametric form $\vec{r}=\vec{a}+t\vec{b}$ where $t\in R$

$$\vec{r}=(\hat{i}+2\hat{j}-3\hat{k})+t(4\hat{i}+5\hat{j}-7\hat{k})$$

(ii) vector equation in non − parametric form $(\vec{r}-\vec{a})\times\vec{b}=0$

$$[\vec{r}-(\hat{i}+2\hat{j}-3\hat{k})]\times(4\hat{i}+5\hat{j}-7\hat{k})=0$$

(iii) Cartesian equations of the straight line.

$$\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$$

Here $(x_1,y_1,z_1)=(1,2,-3)$ & $(b_1,b_2,b_3)=(4,5,-7)$

$$\frac{x-1}{4}=\frac{y-2}{5}=\frac{z+3}{-7}$$

Example 6.25: The vector equation in parametric form of a line is $\vec{r}=(3\hat{i}-2\hat{j}+6\hat{k})+t(2\hat{i}-\hat{j}+3\hat{k})$. Find (i) the direction cosines of the straight line (ii) vector equation in non − parametric form of the line (iii) Cartesian equations of the line.

$$\vec{r}=(3\hat{i}-2\hat{j}+6\hat{k})+t(2\hat{i}-\hat{j}+3\hat{k}).$$

Comparing with $\vec{r}=\vec{a}+t\vec{b}$

Let $\vec{a}=3\hat{i}-2\hat{j}+6\hat{k}$ and $\vec{b}=2\hat{i}-\hat{j}+3\hat{k}$

(i) the direction cosines of the straight line

$$\vec{b}=2\hat{i}-\hat{j}+3\hat{k}$$

$$|\vec{b}|=\sqrt{2^2+(-1)^2+3^2}=\sqrt{4+1+9}$$

$$|\vec{b}|=\sqrt{14}$$

Here $(b_1,b_2,b_3)=(2,-1,3)$

$$\text{Direction cosines}=\left(\frac{b_1}{|\vec{b}|},\frac{b_2}{|\vec{b}|},\frac{b_3}{|\vec{b}|}\right)=\left(\frac{2}{\sqrt{14}},\frac{-1}{\sqrt{14}},\frac{3}{\sqrt{14}}\right)$$

(ii) vector equation in non − parametric form : $(\vec{r}-\vec{a})\times\vec{b}=0$

$$\left(\vec{r}-(3\hat{i}-2\hat{j}+6\hat{k})\right)\times(2\hat{i}-\hat{j}+3\hat{k})=0$$

(iii) Cartesian equations of the straight line: $\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$

Here $(x_1,y_1,z_1)=(3,-2,6)$ & $(b_1,b_2,b_3)=(2,-1,3)$

Example 6.26: Find the vector equation in parametric form and Cartesian equations of the line passing through $(-4,2,-3)$ and is parallel to the line $\frac{-x-2}{4}=\frac{y+3}{-2}=\frac{2z-6}{3}$.

Let $\vec{a}=-4\hat{\imath}+2\hat{\jmath}-3\hat{k}$

$$\frac{-x-2}{4}=\frac{y+3}{-2}=\frac{2z-6}{3}\Rightarrow\frac{-(x+2)}{4}=\frac{y+3}{-2}=\frac{2(z-3)}{3}$$

$$\frac{x+2}{-4}=\frac{y+3}{-2}=\frac{z-3}{\frac{3}{2}}$$

$$\vec{b}=-4\hat{\imath}-2\hat{\jmath}+\frac{3}{2}\hat{k}\quad\Rightarrow\vec{b}=-\frac{1}{2}(8\hat{\imath}+4\hat{\jmath}-3\hat{k})$$

$$\boxed{\vec{a}=\lambda\vec{b}}$$

$\vec{b}$ *is parallel to the vector* $8\hat{\imath}+4\hat{\jmath}-3\hat{k}$ *. Hence* $\vec{b}=8\hat{\imath}+4\hat{\jmath}-3\hat{k}$

vector equation in parametric form $\vec{r}=\vec{a}+t\vec{b}$ *where* $t\in R$

$$\vec{r}=(-4\hat{\imath}+2\hat{\jmath}-3\hat{k})+t(8\hat{\imath}+4\hat{\jmath}-3\hat{k})$$

cartesian equation is $\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}\quad\Rightarrow\frac{x+4}{8}=\frac{y-2}{4}=\frac{z+3}{-3}$

Example 6.27: Find the vector equation in parametric form and Cartesian equations of a straight passing through the points $(-5,7,-4)$ and $(13,-5,2)$. Find the point where the straighyt line crossses the xy − plane.

vector equation in parametric form $\vec{r}=\vec{a}+t(\vec{b}-\vec{a})$

Here $\vec{a}=-5\hat{\imath}+7\hat{\jmath}-4\hat{k},\vec{b}=13\hat{\imath}-5\hat{\jmath}+2\hat{k}$

$$\vec{b}-\vec{a}=13\hat{\imath}-5\hat{\jmath}+2\hat{k}+5\hat{\imath}-7\hat{\jmath}+4\hat{k}\Rightarrow\boxed{\vec{b}-\vec{a}=18\hat{\imath}-12\hat{\jmath}+6\hat{k}}$$

$$\vec{r}=-5\hat{\imath}+7\hat{\jmath}-4\hat{k}+t(18\hat{\imath}-12\hat{\jmath}+6\hat{k})$$

Cartesian equation:

Here $(x_1,y_1,z_1)=(-5,7,-4)$ *and* $(x_2,y_2,z_2)=(13,-5,2)$.

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}\Rightarrow\frac{x+5}{\cancel{18}\ 3}=\frac{y-7}{\cancel{-12}\ -2}=\frac{z+4}{\cancel{6}\ 1}$$

$$\frac{x+5}{3}=\frac{y-7}{-2}=\frac{z+4}{1}$$

An arbitrary point on the straight line is of the form

$$\frac{x+5}{3}=\frac{y-7}{-2}=\frac{z+4}{1}=t$$

$$\frac{x+5}{3}=t,\frac{y-7}{-2}=t,\frac{z+4}{1}=t\Rightarrow\boxed{\frac{x-3}{2}=\frac{y+2}{-1}=\frac{z-6}{3}}$$

$x = 3t - 5, y = -2t + 7,\ z = t - 4 \implies (3t - 5, -2t + 7, t - 4)$

Since the straight line crosses the xy − plane, i.e $z = 0$

$\therefore t - 4 = 0 \implies t = 4$

sub $t = 4$ *in* $(3t - 5, -2t + 7, t - 4)$

$(3(4) - 5, -2(4) + 7, 4 - 4) = (12 - 5, -8 + 7, 0) = (7, -1, 0)$

Hence the straight line crosses the xy − plane at $(7, -1, 0)$

Example 6.28: Find the angle between the straight line $\dfrac{x+3}{2} = \dfrac{y-1}{2} = -z$ **with coordinate axes.**

$$\frac{x+3}{2} = \frac{y-1}{2} = -z \implies \frac{x+3}{2} = \frac{y-1}{2} = \frac{z}{-1}$$

As the angle between the given straight line with the coordinate axes are same as the angles made by $\vec{b}$ *with the coordinate axes,*

$$\text{Let } \vec{b} = 2\hat{\imath} + 2\hat{\jmath} - \hat{k}$$

$$|\vec{b}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{4 + 4 + 1}$$

$$|\vec{b}| = \sqrt{9} \implies |\vec{b}| = 3$$

$$\text{Direction cosines } = \left(\frac{b_1}{|\vec{b}|}, \frac{b_2}{|\vec{b}|}, \frac{b_3}{|\vec{b}|}\right) = \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$$

$$\text{Direction cosines } = \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right) \implies (\cos\alpha, \cos\beta, \cos\gamma) = \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$$

Where α, β, γ *are the angles made by* $\hat{b}$ *with the positive x − axis, positive y − axis, and positive z − axis*

$$\cos\alpha = \frac{2}{3}, \quad \cos\beta = \frac{2}{3}, \cos\gamma = -\frac{1}{3}$$

$$\alpha = \cos^{-1}\left(\frac{2}{3}\right), \beta = \cos^{-1}\left(\frac{2}{3}\right), \ \gamma = \cos^{-1}\left(\frac{-1}{3}\right)$$

Example 6.29: Find the angle between the lines $\vec{r} = (\hat{\imath} + 2\hat{\jmath} + 4\hat{k}) + t(2\hat{\imath} + 2\hat{\jmath} + \hat{k})$ **and the straight line passing through the points** $(5, 1, 4)$ **and** $(9, 2, 12)$.

$$\vec{r} = (\hat{\imath} + 2\hat{\jmath} + 4\hat{k}) + t(2\hat{\imath} + 2\hat{\jmath} + \hat{k})$$

parallel to the vector is $2\hat{\imath} + 2\hat{\jmath} + \hat{k}$

Equation of the straight line passing through the points $(5, 1, 4)$ *and* $(9, 2, 12)$.

vector equation in parametric form $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

$$\text{Here } \vec{a} = 5\hat{\imath} + \hat{\jmath} + 4\hat{k}, \vec{b} = 9\hat{\imath} + 2\hat{\jmath} + 12\hat{k}$$

$$\vec{b} - \vec{a} = 9\hat{\imath} + 2\hat{\jmath} + 12\hat{k} - 5\hat{\imath} - \hat{\jmath} - 4\hat{k}$$

$$\vec{b}-\vec{a}=4\hat{\imath}+\hat{\jmath}+8\hat{k}$$

parallel to the vector is $4\hat{\imath}+\hat{\jmath}+8\hat{k}$

The angle between the given two straight lines is $\cos\theta=\dfrac{\vec{b}.\vec{d}}{|\vec{b}||\vec{d}|}$

where $\vec{b}=2\hat{\imath}+2\hat{\jmath}+\hat{k}$ *and* $\vec{d}=4\hat{\imath}+\hat{\jmath}+8\hat{k}$

$$cos\theta=\frac{(2\hat{\imath}+2\hat{\jmath}+\hat{k}).(4\hat{\imath}+\hat{\jmath}+8\hat{k})}{|2\hat{\imath}+2\hat{\jmath}+\hat{k}||4\hat{\imath}+\hat{\jmath}+8\hat{k}|}\Rightarrow cos\theta=\frac{8+2+8}{\sqrt{2^2+2^2+1^2}\sqrt{4^2+1^2+8^2}}$$

$$cos\theta=\left(\frac{18}{\sqrt{4+4+1}\sqrt{16+1+64}}\right)\Rightarrow\theta=cos^{-1}\left(\frac{18}{\sqrt{9}\sqrt{81}}\right)$$

$$\theta=cos^{-1}\left(\frac{18}{3\times 9}\right)\Rightarrow\theta=cos^{-1}\left(\frac{2}{3}\right)$$

Example 6.30*: *Find the angle between the straight lines $\dfrac{x-4}{2}=\dfrac{y}{1}=\dfrac{z+1}{-2}$ ***and*** $\dfrac{x-1}{4}=\dfrac{y+1}{-4}=\dfrac{z-2}{2}$ ***and state whether they are parallel or perpendicular.***

$$\frac{x-4}{2}=\frac{y}{1}=\frac{z+1}{-2}\ and\ \frac{x-1}{4}=\frac{y+1}{-4}=\frac{z-2}{2}$$

Here $\vec{b}=2\hat{\imath}+\hat{\jmath}-2\hat{k}$ *and* $\vec{d}=4\hat{\imath}-4\hat{\jmath}+2\hat{k}$

The angle between the given two straight lines is $\cos\theta=\dfrac{\vec{b}.\vec{d}}{|\vec{b}||\vec{d}|}$

$$cos\theta=\frac{(2\hat{\imath}+\hat{\jmath}-2\hat{k}).(4\hat{\imath}-4\hat{\jmath}+2\hat{k})}{|2\hat{\imath}+\hat{\jmath}-2\hat{k}||4\hat{\imath}-4\hat{\jmath}+2\hat{k}|}$$

$$cos\theta=\frac{8-4-4}{\sqrt{2^2+(1)^2+(-2)^2}\sqrt{4^2+(-4)^2+2^2}}$$

$$cos\theta=\frac{8-4-4}{\sqrt{4+1+4}\sqrt{16+16+4}}\Rightarrow cos\theta=\frac{0}{\sqrt{9}\sqrt{36}}$$

$$cos\theta=0\Rightarrow\theta=\frac{\pi}{2}$$

Thus the two straight lines are perpendicular.

Example 6.31*: *Show that the straight line passing through the points $A(6,7,5)$ ***and*** $B(8,10,6)$ ***is perpendicular to the straight line passing through the points*** $C(10,2,-5)$ ***and*** $D(8,3,-4)$.

$$\overrightarrow{OA}=6\hat{\imath}+7\hat{\jmath}+5\hat{k},\overrightarrow{OB}=8\hat{\imath}+10\hat{\jmath}+6\hat{k}$$

$$\vec{b}=\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=8\hat{\imath}+10\hat{\jmath}+6\hat{k}-(6\hat{\imath}+7\hat{\jmath}+5\hat{k})$$

$$=8\hat{\imath}+10\hat{\jmath}+6\hat{k}-6\hat{\imath}-7\hat{\jmath}-5\hat{k}$$

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$$\boxed{\vec{b} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}}$$

$$\overrightarrow{OC} = 10\hat{\imath} + 2\hat{\jmath} - 5\hat{k}, \overrightarrow{OD} = 8\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$$

$$\vec{d} = \overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = 8\hat{\imath} + 3\hat{\jmath} - 4\hat{k} - \left(10\hat{\imath} + 2\hat{\jmath} - 5\hat{k}\right)$$

$$= 8\hat{\imath} + 3\hat{\jmath} - 4\hat{k} - 10\hat{\imath} - 2\hat{\jmath} + 5\hat{k}$$

$$\vec{d} = -2\hat{\imath} + \hat{\jmath} + \hat{k}$$

$$\vec{b}.\vec{d} = \left(2\hat{\imath} + 3\hat{\jmath} + \hat{k}\right).\left(-2\hat{\imath} + \hat{\jmath} + \hat{k}\right) = -4 + 3 + 1 = 0$$

The two vectors are perpendicular, and hence the two straight lines are perpendicular.

Example 6.32: Show that the lines $\dfrac{x-1}{4} = \dfrac{2-y}{6} = \dfrac{z-4}{12}$ ***and*** $\dfrac{x-3}{-2} = \dfrac{y-3}{3} = \dfrac{5-z}{6}$ ***are parallel.***

$$\frac{x-1}{4} = \frac{y-2}{-6} = \frac{z-4}{12} \text{ and } \frac{x-3}{-2} = \frac{y-3}{3} = \frac{z-5}{-6}$$

The straight line $\dfrac{x-1}{4} = \dfrac{y-2}{-6} = \dfrac{z-4}{12}$ *is parallel to the vector* $4\hat{\imath} - 6\hat{\jmath} + 12\hat{k}$ *and the straight line* $\dfrac{x-3}{-2} = \dfrac{y-3}{3} = \dfrac{5-z}{6}$ *is parallel to the vector* $-2\hat{\imath} + 3\hat{\jmath} - 6\hat{k}$

$$\text{since } 4\hat{\imath} - 6\hat{\jmath} + 12\hat{k} = -2\left(-2\hat{\imath} + 3\hat{\jmath} - 6\hat{k}\right)$$

The two vectors are parallel, and hence the two straight lines are parallel.

1. *Find the non − parametric form of vector equation and cartesian equations of the straight line passing through the point with position vector* $\mathbf{4\hat{\imath} + 3\hat{\jmath} - 7\hat{k}}$ ***and parallel to the vector*** $\mathbf{2\hat{\imath} - 6\hat{\jmath} + 7\hat{k}}$

Let $\vec{a} = 4\hat{\imath} + 3\hat{\jmath} - 7\hat{k}$ *and* $\vec{b} = 2\hat{\imath} - 6\hat{\jmath} + 7\hat{k}$

Non − parametric form of vector equation is $(\vec{r} - \vec{a}) \times \vec{b} = 0$

$$\left(\vec{r} - \left(4\hat{\imath} + 3\hat{\jmath} - 7\hat{k}\right)\right) \times (2\hat{\imath} - 6\hat{\jmath} + 7\hat{k}) = 0$$

cartesian equation is $\dfrac{x-x_1}{b_1} = \dfrac{y-y_1}{b_2} = \dfrac{z-z_1}{b_3}$

$$\because (x_1, y_1, z_1) = (4,3,7) \text{ and } (b_1, b_2, b_3) = (2,-6,7)$$

$$\frac{x-4}{2} = \frac{y-3}{-6} = \frac{z+7}{7}$$

2. *Find the parametric form of vector equation and cartesian equation of the straight line passing thorugh the point* $(-2,3,4)$ ***and parallel to the straight line*** $\frac{x-1}{-4}=\frac{y+3}{5}=\frac{8-z}{6}$

$$\frac{x-1}{-4}=\frac{y+3}{5}=\frac{8-z}{6} \Longrightarrow \frac{x-1}{-4}=\frac{y+3}{5}=\frac{z-8}{-6}$$

Let $\vec{a}=-2\hat{\imath}+3\hat{\jmath}+4\hat{k}$ *and* $\vec{b}=-4\hat{\imath}+5\hat{\jmath}-6\hat{k}$

Parametric vector equation of a straight is $\vec{r}=\vec{a}+t\vec{b}$ *where* $t\in R$

$$\vec{r}=\left(-2\hat{\imath}+3\hat{\jmath}+4\hat{k}\right)+t\left(-4\hat{\imath}+5\hat{\jmath}-6\hat{k}\right)$$

cartesian equation is $\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$

$\because (x_1,y_1,z_1)=(-2,3,4)$ *and* $(b_1,b_2,b_3)=(-4,5,-6)$

$$\frac{x+2}{-4}=\frac{y-3}{5}=\frac{z-4}{-6}$$

3. *Find the points where the straight line passes through* $(6,7,4)$ ***and*** $(8,4,9)$ ***cuts the xz and yz planes.***

Cartesian equation: *Here* $(x_1,y_1,z_1)=(6,7,4)$ *and* $(x_2,y_2,z_2)=(8,4,9)$

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \Longrightarrow \frac{x-6}{8-6}=\frac{y-7}{4-7}=\frac{z-4}{9-4}$$

$$\boxed{\frac{x-6}{2}=\frac{y-7}{-3}=\frac{z-4}{5}}$$

An arbitrary point on the straight line is of the form

$$\frac{x-6}{2}=\frac{y-7}{-3}=\frac{z-4}{5}=t \Longrightarrow \frac{x-6}{2}=t,\frac{y-7}{-3}=t,\frac{z-4}{5}=t$$

$$x=2t+6, y=-3t+7,\ z=5t+4 \Longrightarrow (2t+6,-3t+7,5t+4)$$

$\therefore$ *Point on the line is* $(2t+6,-3t+7,5t+4)$ ... (1)

To find the point that the line cut xz plane i.e $y=0$

$$\therefore -3t+7=0 \Longrightarrow -3t=-7 \Longrightarrow t=\frac{7}{3}$$

Sub $t=\frac{7}{3}$ *in* (1) $\left(2\left(\frac{7}{3}\right)+6,-3\left(\frac{7}{3}\right)+7,5\left(\frac{7}{3}\right)+4\right)$

$$=\left(2\left(\frac{7}{3}\right)+6,-3\left(\frac{7}{3}\right)+7,5\left(\frac{7}{3}\right)+4\right)=\left(\frac{14}{3}+6,-7+7,\frac{35}{3}+4\right)$$

$$=\left(\frac{14+18}{3},0,\frac{35+12}{3}\right)$$

Hence the straight line cut the xz − plane at $\left(\frac{32}{3},0,\frac{47}{3}\right)$

To find the point that the line cut yz plane i.e $x = 0$

$$\therefore 2t + 6 = 0 \implies 2t = -6 \implies t = -3$$

Sub $t = -3$ in (1) $(2t + 6, -3t + 7, 5t + 4)$

$$= (2(-3) + 6, -3(-3) + 7, 5(-3) + 4) = (-6 + 6, 9 + 7, -15 + 4)$$

$$= (0, 16, -11)$$

Hence the straight line cut the $yz-$plane at $(0, 16, -11)$

Let $\vec{b} = 5\hat{\imath} + 6\hat{\jmath} + 7\hat{k}$ and $\vec{a} = 7\hat{\imath} + 9\hat{\jmath} + 13\hat{k}$

The parametric form of vector equation of a straight line passing through two points $\vec{a}$ and $\vec{b}$ is

$$\vec{r} = \vec{a} + t(\vec{b} - \vec{a}), t \in R$$

$$\therefore \vec{r} = (7\hat{\imath} + 9\hat{\jmath} + 13\hat{k}) + t\left((7-5)\hat{\imath} + (9-6)\hat{\jmath} + (13-7)\hat{k}\right)$$

$$\vec{r} = (7\hat{\imath} + 9\hat{\jmath} + 13\hat{k}) + t(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}), t \in R$$

The cartesian of the straight line passing through two points is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

4. Find the direction cosines of the straight line passing through the points $(5, 6, 7)$ and $(7, 9, 13)$. Also, find the parametric form of vector and cartesian equations of the straight line passing through two given points

vector equation in parametric form $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

Here $\vec{a} = 5\hat{\imath} + 6\hat{\jmath} + 7\hat{k}, \vec{b} = 7\hat{\imath} + 9\hat{\jmath} + 13\hat{k}$

$$\vec{b} - \vec{a} = 7\hat{\imath} + 9\hat{\jmath} + 13\hat{k} - 5\hat{\imath} - 6\hat{\jmath} - 7\hat{k}$$

$$\vec{b} - \vec{a} = 2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$$

$$\vec{r} = 5\hat{\imath} + 6\hat{\jmath} + 7\hat{k} + t(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k})$$

Cartesian equation:

Here $(x_1, y_1, z_1) = (5, 6, 7)$ and $(x_2, y_2, z_2) = (7, 9, 13)$.

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \implies \frac{x - 5}{2} = \frac{y - 6}{3} = \frac{z - 7}{6}$$

Direction cosines of given straight line is same as the direction cosines of $\vec{b}$.

Let $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$

$$|\vec{b}| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{4 + 9 + 36}$$

$$|\vec{b}| = \sqrt{49} \implies |\vec{b}| = 7$$

$$\text{Direction cosines} = \left(\frac{b_1}{|\vec{b}|}, \frac{b_2}{|\vec{b}|}, \frac{b_3}{|\vec{b}|}\right) = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$$

$$\text{Direction cosines} = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$$

5. *Find the angle between the following lines*:

(i) $\vec{r} = (4\hat{i} - \hat{j}) + t(\hat{i} + 2\hat{j} - 2\hat{k}), \vec{r} = (\hat{i} - 2\hat{j} + 4\hat{k}) + s(-\hat{i} - 2\hat{j} + 2\hat{k})$

(ii) $\frac{x+4}{3} = \frac{y-7}{4} = \frac{z+5}{5}, \vec{r} = (4\hat{k}) + t(2\hat{i} + \hat{j} + \hat{k})$

(iii) $2x = 3y = -2$ *and* $6x = -y - 4z$

(i) *Given lines are*

$\vec{r} = (4\hat{i} - \hat{j}) + t(\hat{i} + 2\hat{j} - 2\hat{k})$

$\therefore$ *parallel to the vector is* $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

and $\vec{r} = (\hat{i} - 2\hat{j} + 4\hat{k}) + s(-\hat{i} - 2\hat{j} + 2\hat{k})$

$\therefore$ *parallel to the vector is* $\vec{d} = -\hat{i} - 2\hat{j} + 2\hat{k}$

Let θ *be the angle between the given lines*

$$\cos\theta = \frac{\vec{b}.\vec{d}}{|\vec{b}||\vec{d}|}$$

$$\cos\theta = \frac{(\hat{i} + 2\hat{j} - 2\hat{k}).(-\hat{i} - 2\hat{j} + 2\hat{k})}{\sqrt{1^2 + 2^2 + (-2)^2}\sqrt{(-1)^2 + (-2)^2 + (2)^2}}$$

$$\cos\theta = \frac{-1 - 4 - 4}{\sqrt{1+4+4}\sqrt{1+4+4}} = \frac{-9}{\sqrt{9}\sqrt{9}} = \frac{-9}{9}$$

$$\cos\theta = -1 \Longrightarrow \theta = \cos^{-1}(-1)$$

$$\theta = \pi$$

(ii) $\frac{x+4}{3} = \frac{y-7}{4} = \frac{z+5}{5}, \vec{r} = (4\hat{k}) + t(2\hat{i} + \hat{j} + \hat{k})$

(ii) *Given lines are* $\frac{x+4}{3} = \frac{y-7}{4} = \frac{z+5}{5}$

parallel to the vector is $\vec{b} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

and $\vec{r} = (4\hat{k}) + t(2\hat{i} + \hat{j} + \hat{k})$ *parallel to the vector is* $\vec{d} = 2\hat{i} + \hat{j} + \hat{k}$

$$\boxed{\therefore \cos\theta = \frac{\vec{b}.\vec{d}}{|\vec{b}||\vec{d}|}}$$

$$\cos\theta = \frac{(3\hat{i}+4\hat{j}+5\hat{k})(2\hat{i}+\hat{j}+\hat{k})}{\sqrt{3^2+4^2+5^2}\sqrt{2^2+1^2+1^2}} = \frac{6+4+5}{\sqrt{9+16+25}\sqrt{4+1+1}}$$

$$\cos\theta = \frac{6+4+5}{\sqrt{50}\sqrt{6}} \Rightarrow \cos\theta = \frac{15}{5\sqrt{2}\sqrt{6}} \Rightarrow \cos\theta = \frac{3}{2\sqrt{3}}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{3}{2\sqrt{3}}\right)$$

(iii) Given lines are $2x = 3y = -z$ and $6x = -y = -4z$

$$2x = 3y = -z \Rightarrow \frac{x-0}{\frac{1}{2}} = \frac{y-0}{\frac{1}{3}} = \frac{z-0}{-1}$$

$$\vec{b} = \frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} - \hat{k}$$

$$6x = -y = -4z \Rightarrow \frac{x-0}{\frac{1}{6}} = \frac{y-0}{\frac{1}{-1}} = \frac{z-0}{-\frac{1}{4}}$$

$$\vec{d} = \frac{1}{6}\hat{i} - \hat{j} - \frac{1}{4}\hat{k}$$

$$\vec{b}.\vec{d} = \left(\frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} - \hat{k}\right).\left(\frac{1}{6}\hat{i} - \hat{j} - \frac{1}{4}\hat{k}\right)$$

$$= \frac{1}{12} - \frac{1}{3} + \frac{1}{4} = \frac{1-4+3}{12} = 0$$

$\therefore \vec{b} \perp \vec{d}$ $\therefore$ *the angle between* $\vec{b}$ and $\vec{d}$ *is* $\frac{\pi}{2}$

6. The vertices of ΔABC are $A(7,2,1), B(6,0,3)$ and $C(4,2,0)$, Find $\angle ABC$

Given : $A(7,2,1), B(6,0,3)$ *and* $C(4,2,4)$

$\angle ABC$ *is lies between* AB *and* BC

i.e $\overrightarrow{OA} = 7\hat{i} + 2\hat{j} + \hat{k}, \overrightarrow{OB} = 6\hat{i} + 0\hat{j} + 3\hat{k}, \overrightarrow{OC} = 4\hat{i} + 2\hat{j} + 4\hat{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= 6\hat{i} + 0\hat{j} + 3\hat{k} - (7\hat{i} + 2\hat{j} + \hat{k})$$
$$= 6\hat{i} + 0\hat{j} + 3\hat{k} - 7\hat{i} - 2\hat{j} - \hat{k}$$
$$\overrightarrow{AB} = -\hat{i} - 2\hat{j} + 2\hat{k}$$
$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$
$$= 4\hat{i} + 2\hat{j} + 4\hat{k} - (6\hat{i} + 0\hat{j} + 3\hat{k})$$
$$= 4\hat{i} + 2\hat{j} + 4\hat{k} - 6\hat{i} - 0\hat{j} - 3\hat{k}$$
$$\overrightarrow{BC} = -2\hat{i} + 2\hat{j} + \hat{k}$$

$$\therefore \cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{BC}}{|\overrightarrow{AB}||\overrightarrow{BC}|}$$

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Here $\overrightarrow{AB} = -\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ and $\overrightarrow{BC} = -2\hat{\imath} + 2\hat{\jmath} + \hat{k}$

$$\cos\theta = \frac{(-\hat{\imath} - 2\hat{\jmath} + 2\hat{k}).(-2\hat{\imath} + 2\hat{\jmath} + \hat{k})}{\sqrt{(-1)^2 + (-1)^2 + 2^2}\sqrt{2^2 + (-2)^2 + 3^2}} = \frac{2 - 4 + 2}{\sqrt{9 + 16 + 25}\sqrt{4 + 1 + 1}}$$

$$\cos\theta = \frac{0}{\sqrt{25}\sqrt{6}} \implies \cos\theta = 0 \implies \theta = 90^\circ$$

$$\boxed{\angle ABC = 90^\circ}$$

7. If the striaght line joining the points $(2,1,4)$ and $(a-1,4,-1)$ is parallel to $(0,2,b-1)$ and $(5,3,-2)$. find the values of a and b

Cartesian equation of the straight line passing through two points $(2,1,4)$ and $(a-1,4,-1)$

$\because (x_1,y_1,z_1) = (2,1,4), (x_2,y_2,z_2) = (a-1,4,-1)$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \implies \frac{x-2}{a-1-2} = \frac{y-1}{4-1} = \frac{z-4}{-1-4}$$

$$\frac{x-2}{a-3} = \frac{y-1}{3} = \frac{z-4}{-5} \quad \dots(1)$$

parallel to the vector is $\vec{b} = (a-3)\hat{\imath} + 3\hat{\jmath} - 5\hat{k}$

Similarly cartesian equation of straight lines passing through two points $(5,3,-2)$ and $(0,2,b-1)$ is

$\because (x_1,y_1,z_1) = (5,3,-2), (x_2,y_2,z_2) = (0,2,b-1)$

$$\frac{x-5}{0-5} = \frac{y-3}{2-3} = \frac{z+2}{b-1+2} \implies \frac{x-5}{-5} = \frac{y-3}{-1} = \frac{z+2}{b+1} \dots(2)$$

parallel to the vector is $\vec{d} = -5\hat{\imath} - \hat{\jmath} + (b+1)\hat{k}$

(1) and (2) are parallel

$$\vec{b} = (a-3)\hat{\imath} + 3\hat{\jmath} - 5\hat{k} \implies \vec{d} = -5\hat{\imath} - \hat{\jmath} + (b+1)\hat{k}$$

$$\vec{b} = -3\vec{d}$$

$$(a-3)\hat{\imath} + 3\hat{\jmath} - 5\hat{k} = -3(-5\hat{\imath} - \hat{\jmath} + (b+1)\hat{k})$$

$$(a-3)\hat{\imath} + 3\hat{\jmath} - 5\hat{k} = 15\hat{\imath} + 3\hat{\jmath} - 3(b+1)\hat{k}$$

Equating the coefficient of $\hat{\imath}, \hat{k}$

$$\therefore a - 3 = 15 \implies a = 15 + 3$$

$$\boxed{a = 18}$$

$$3(b+1) = 5 \implies b + 1 = \frac{5}{3}$$

$$b = \frac{5}{3} - 1 \implies b = \frac{5-3}{3} \implies b = \frac{2}{3}$$

$$\therefore a = 18 \text{ and } b = \frac{2}{3}$$

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8. If the straight lines $\frac{x-5}{5m+2}=\frac{2-y}{5}=\frac{1-z}{-1}$ and $x=\frac{2y+1}{4m}=\frac{1-z}{-3}$ are perpendicular to each other, find the value of m

$$\frac{x-5}{5m+2}=\frac{2-y}{5}=\frac{1-z}{-1} \Longrightarrow \frac{x-5}{5m+2}=\frac{y-2}{-5}=\frac{z-1}{1}$$

$$\therefore \vec{b}=(5m+2)\hat{\imath}-5\hat{\jmath}+\hat{k}$$

$$and\ x=\frac{2y+1}{4m}=\frac{1-z}{-3} \Longrightarrow x=\frac{\frac{2y+1}{2}}{\frac{4m}{2}}=\frac{1-z}{-3} \Longrightarrow \frac{x}{1}=\frac{y+\frac{1}{2}}{2m}=\frac{z-1}{3}$$

$$\therefore \vec{d}=\hat{\imath}+2m\hat{\jmath}+3\hat{k}$$

$$Since\ \vec{b}\perp\vec{d} \Rightarrow \vec{b}.\vec{d}=0$$

$$\left((5m+2)\hat{\imath}-5\hat{\jmath}+\hat{k}\right).\left(\hat{\imath}+2m\hat{\jmath}+3\hat{k}\right)=0$$

$$(5m+2)1+2m(-5)+3(1)=0$$

$$5m+2-10m+3=0 \Longrightarrow 5-5m=0$$

$$5=5m \Longrightarrow \boxed{m=1}$$

9. Show that the points $(2,3,4),(-1,4,5)$ and $(8,1,2)$ are collinear

Let the points $A(2,3,4), B(-1,4,5)$ *and* $C(8,1,2)$

Equation of the straight line AB is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

Here $(x_1, y_1, z_1)=(2,3,4)$ *and* $(x_2, y_2, z_2)=(-1,4,5)$

$$\frac{x-2}{-1-2}=\frac{y-3}{4-3}=\frac{z-4}{5-4} \Longrightarrow \frac{x-2}{-3}=\frac{y-3}{1}=\frac{z-4}{1}$$

Substitute the point $C(8,1,2)$ *in line* $\frac{x-2}{-3}=\frac{y-3}{1}=\frac{z-4}{1}$

$$\frac{8-2}{-3}=\frac{1-3}{1}=\frac{2-4}{1}$$

Since the point C satisfies the equation of the line joining A and B all the three lie on the same line

Hence the points are collinear

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EXERCISE 6.5

Example 6.33: Find the point of intersection of the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ **and** $\frac{x-4}{5}=\frac{y-1}{2}=z.$

Given lines: $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ *and* $\frac{x-4}{5}=\frac{y-1}{2}=z$

Take: $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=s \Rightarrow \frac{x-1}{2}=s, \frac{y-2}{3}=s, \frac{z-3}{4}=s$

$x-1=2s,\ y-2=3s,\ z-3=4s \Rightarrow x=2s+1,\ y=3s+2,\ z=4s+3$

Any point is of the form : $(2s+1, 3s+2, 4s+3)$

Take: $\frac{x-4}{5}=\frac{y-1}{2}=z=t \Rightarrow \frac{x-4}{5}=t, \frac{y-1}{2}=t, z=t$

$x-4=5t,\ y-1=2t,\ z=t \Rightarrow x=5t+4,\ y=2t+1,\ z=t$

Any point is of the form $(5t+4, 2t+1, t)$

since the lines are intersecting

$(2s+1, 3s+2, 4s+3)=(5t+4, 2t+1, t)$

$2s+1=5t+4 \Rightarrow 2s-5t=4-1 \Rightarrow 2s-5t=3 \ldots (1)$

$3s+2=2t+1 \Rightarrow 3s-2t=1-2 \Rightarrow 3s-2t=-1 \ldots (2)$

Solve (1) & (2)

$(1)\times 2 \Rightarrow 4s-10t=6$

$\quad (-)\ \ (+)\ \ \ (+)$

$(2)\times 5 \Rightarrow 15s-10t=-5$

$-11s=11 \Rightarrow s=-\frac{11}{11} \Rightarrow s=-1$

subs $s=-1$ *in* (1) $2s-5t=3$

$2(-1)-5t=3 \Rightarrow -2-5t=3 \Rightarrow -5t=3+2 \Rightarrow -5t=5$

$t=-\frac{5}{5} \Rightarrow t=-1$

$s=-1$ *in* $(2s+1, 3s+2, 4s+3)=(2(-1)+1, 3(-1)+2, 4(-1)+3)$

$=(-2+1, -3+2, -4+3)$

The point of intersection is $(-1,-1,-1)$

Example 6.34: Find the equation of a straight line passing through the point of intersection of the straight lines $\vec{r}=(\hat{\imath}+3\hat{\jmath}-\hat{k})+t(2\hat{\imath}+3\hat{\jmath}+2\hat{k})$ **and** $\frac{x-2}{1}=\frac{y-4}{2}=\frac{z+3}{4}$**, and perpendicular to both straight lines.**

The Cartesian equations of the straight line

$\vec{r}=(\hat{\imath}+3\hat{\jmath}-\hat{k})+t(2\hat{\imath}+3\hat{\jmath}+2\hat{k})$

$$\frac{x-1}{2}=\frac{y-3}{3}=\frac{z+1}{2}=s \implies \frac{x-1}{2}=s, \frac{y-3}{3}=s, \frac{z+1}{2}=s$$

$x-1=2s,\ y-3=3s,\ z+1=2s \implies x=2s+1,\ y=3s+3,\ z=2s-1$

Any point on this line is of the form $(2s+1, 3s+3, 2s-1)$

$$Take: \frac{x-2}{1}=\frac{y-4}{2}=\frac{z+3}{4}=t \implies \frac{x-2}{1}=t, \frac{y-4}{2}=t, \frac{z+3}{4}=t$$

$x-2=t,\quad y-4=2t,\ z+3=4t \implies x=t+2,\quad y=2t+4,\ z=4t-3$

Any point on this line is of the form $(t+2, 2t+4, 4t-3)$

since given lines intersect,

$(2s+1, 3s+3, 2s-1)=(t+2, 2t+4, 4t-3)$

Equating the coordinates of x *and* y

$2s+1=t+2 \implies 2s-t=2-1$

$2s-t=1 \ldots (1)$

$3s+3=2t+4 \implies 3s-2t=4-3$

$3s-2t=1 \ \ldots\ (2)$

Solve (1) & (2)

$(1)\times 2 \implies 4s-2t=2$

$\quad (-)\ (+)\quad (-)$

$(2) \implies 3s-2t=1$

$s=1$

subs $s=1$ *in* (1) $2s-t=1$

$2(1)-t=1 \implies 2-t=1$

$-t=1-2 \implies -t=-1 \implies t=1$

$t=1\ in (t+2, 2t+4, 4t-3)$

$=(1+2, 2(1)+4, 4(1)-3)=(3,6,1)$

The point of intersection is $(3,6,1)$

$\vec{r}=(\hat{\imath}+3\hat{\jmath}-\hat{k})+t(2\hat{\imath}+3\hat{\jmath}+2\hat{k}) \implies \vec{b}=2\hat{\imath}+3\hat{\jmath}+2\hat{k}$

$$\frac{x-2}{1}=\frac{y-4}{2}=\frac{z+3}{4} \implies \vec{d}=\hat{\imath}+2\hat{\jmath}+4\hat{k}$$

Since the required line is perpendicular to both $\vec{b}$ *and* $\vec{d}$

$$\therefore\ \vec{b}\times\vec{d}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 3 & 2\\ 1 & 2 & 4\end{vmatrix}=\hat{\imath}(12-4)-\hat{\jmath}(8-2)+\hat{k}(4-3)$$

$\vec{b}\times\vec{d}=8\hat{\imath}-6\hat{\jmath}+\hat{k}$ *is a vector perpendicular to both the given straight lines.*

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Therefore, the required straight line passing through $(3,6,1)$ *and perpendicular to both the given straight lines is the same as the straight line passing through* $(3,6,1)$ *and parallel to* $8\hat{\imath} - 6\hat{\jmath} + \hat{k}$

$\therefore$ *The equation of the required line is* $\vec{r} = \vec{a} + m(\vec{b} \times \vec{d}), m \in R$

$$\vec{r} = (3\hat{\imath} + 6\hat{\jmath} + \hat{k}) + m(8\hat{\imath} - 6\hat{\jmath} + \hat{k})$$

Example 6.35*: *Determine whether the pair of straight lines $\vec{r} = (2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}) + t(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}), \vec{r} = (2\hat{\jmath} - 3\hat{k}) + s(\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$ ***are parallel. Find the shortest distance between them.***

Given two equations $\vec{r} = (2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}) + t(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k})$

and

$$\vec{r} = (2\hat{\imath} - 3\hat{k}) + s(\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$$

$\vec{a} = 2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}, \vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}, \vec{c} = 2\hat{\jmath} - 3\hat{k}, \vec{d} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

Since $\vec{b} \neq \vec{d}$, *they are not parallel*

Shortest distance between the two skew lines $\delta = \dfrac{|(\vec{c} - \vec{a}).(\vec{b} \times \vec{d})|}{|\vec{b} \times \vec{d}|}$

$$\therefore \vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} = \hat{\imath}(9-8) - \hat{\jmath}(6-4) + \hat{k}(4-3)$$

$$\vec{b} \times \vec{d} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$$

$$|\vec{b} \times \vec{d}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{1+4+1} = \sqrt{6}$$

$$|\vec{b} \times \vec{d}| = \sqrt{6}$$

$$\vec{c} - \vec{a} = 2\hat{\jmath} - 3\hat{k} - (2\hat{\imath} + 6\hat{\jmath} + 3\hat{k})$$
$$= 2\hat{\jmath} - 3\hat{k} - 2\hat{\imath} - 6\hat{\jmath} - 3\hat{k}$$
$$\vec{c} - \vec{a} = -2\hat{\imath} - 4\hat{\jmath} - 6\hat{k}$$
$$(\vec{c} - \vec{a}).(\vec{b} \times \vec{d}) = (-2\hat{\imath} - 4\hat{\jmath} - 6\hat{k}).(\hat{\imath} - 2\hat{\jmath} + \hat{k})$$
$$(\vec{c} - \vec{a}).(\vec{b} \times \vec{d}) = -2 + 8 - 6 = 0$$

$\therefore$ *The distance between the two given straight lines is zero.*

Thus, the given lines intersect each other.

Example 6.36*: *Find the shortest distance between the two given straight lines $\vec{r} = (2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) + t(-2\hat{\imath} + \hat{\jmath} - 2\hat{k})$ ***and***

$$\frac{x-3}{2} = \frac{y}{-1} = \frac{z+2}{2}$$

The parametric form of vector equations of the given straight lines are

$$\vec{r} = (2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) + t(-2\hat{\imath} + \hat{\jmath} - 2\hat{k}) \text{ and } \vec{r} = (3\hat{\imath} - 2\hat{k}) + t(2\hat{\imath} - \hat{\jmath} + 2\hat{k})$$

$$\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}, \vec{b} = -2\hat{\imath} + \hat{\jmath} - 2\hat{k}, \vec{c} = 3\hat{\imath} - 2\hat{k}, \vec{d} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$$

$$\vec{b} = -2\hat{\imath} + \hat{\jmath} - 2\hat{k} \Rightarrow \vec{b} = -\left(2\hat{\imath} - \hat{\jmath} + 2\hat{k}\right)$$

$$\vec{b} = -\vec{d}$$

$\vec{b}$ is a scalar multiple of $\vec{d}$, and hence the two straight lines are parallel.

Shortest distance between two parallel straight lines $d = \dfrac{\left|(\vec{c} - \vec{a}) \times \vec{b}\right|}{\left|\vec{b}\right|}$

$$\vec{c} - \vec{a} = \left(3\hat{\imath} - 2\hat{k}\right) - \left(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}\right)$$

$$= 3\hat{\imath} - 2\hat{k} - 2\hat{\imath} - 3\hat{\jmath} - 4\hat{k}$$

$$\vec{c} - \vec{a} = \hat{\imath} - 3\hat{\jmath} - 6\hat{k}$$

$$\therefore (\vec{c} - \vec{a}) \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -3 & -6 \\ -2 & 1 & -2 \end{vmatrix} = \hat{\imath}(6+6) - \hat{\jmath}(-2-12) + \hat{k}(1-6)$$

$$(\vec{c} - \vec{a}) \times \vec{b} = 12\hat{\imath} + 14\hat{\jmath} - 5\hat{k}$$

$$d = \frac{\left|(\vec{c} - \vec{a}) \times \vec{b}\right|}{\left|\vec{b}\right|} \Rightarrow d = \frac{\left|12\hat{\imath} + 14\hat{\jmath} - 5\hat{k}\right|}{\left|-2\hat{\imath} + \hat{\jmath} - 2\hat{k}\right|} \Rightarrow d = \frac{\sqrt{12^2 + 14^2 + (-5)^2}}{\sqrt{(-2)^2 + 1^2 + (-2)^2}}$$

$$d = \frac{\sqrt{144 + 196 + 25}}{\sqrt{4 + 1 + 4}} \Rightarrow d = \frac{\sqrt{365}}{\sqrt{9}} \Rightarrow d = \frac{\sqrt{365}}{3}$$

Example 6.37*: *Find the coordinates of the foot of the perpendicular drawn from the point $(-1, 2, 3)$ to the straight line $\vec{r} = \left(\hat{\imath} - 4\hat{\jmath} + 3\hat{k}\right) + t\left(2\hat{\imath} + 3\hat{\jmath} + \hat{k}\right)$. Also, find the shortest distance from the point to the straight line.

Given equation : $\vec{r} = \left(\hat{\imath} - 4\hat{\jmath} + 3\hat{k}\right) + t\left(2\hat{\imath} + 3\hat{\jmath} + \hat{k}\right)$

$$\vec{a} = \hat{\imath} - 4\hat{\jmath} + 3\hat{k}, \vec{b} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$$

The Cartesian equations of the straight line

$\vec{r} = \left(\hat{\imath} - 4\hat{\jmath} + 3\hat{k}\right) + t\left(2\hat{\imath} + 3\hat{\jmath} + \hat{k}\right)$

$$\frac{x-1}{2} = \frac{y+4}{3} = \frac{z-3}{1}$$

$$\frac{x-1}{2} = \frac{y+4}{3} = \frac{z-3}{1} = t \Rightarrow \frac{x-1}{2} = t, \frac{y+4}{3} = t, \frac{z-3}{1} = t$$

$$x = 2t + 1,\ y = 3t - 4,\ z = t + 3$$

If F is the foot of the perpendicular from to the straight line, then F is of the form $(2t + 1, 3t - 4, t + 3)$

$$\overrightarrow{DF} = \overrightarrow{OF} - \overrightarrow{OD}$$

$$= (2t + 1, 3t - 4, t + 3) - (-1, 2, 3)$$

$$= (2t + 1 + 1, 3t - 4 - 2, t + 3 - 3)$$

$$\overrightarrow{DF} = (2t + 2)\hat{\imath} + (3t - 6)\hat{\jmath} + t\hat{k}$$

Since $\vec{b}$ is perpendicular to $\overrightarrow{DF}$

$\vec{b}.\overrightarrow{DF} = 0 \Rightarrow (2\hat{\imath} + 3\hat{\jmath} + \hat{k}).\left[(2t+2)\hat{\imath} + (3t-6)\hat{\jmath} + t\hat{k}\right] = 0$

$2(2t+2) + 3(3t-6) + 1(t) = 0$

$4t + 4 + 9t - 18 + t = 0 \Rightarrow 14t - 14 = 0$

$14t = 14 \Rightarrow t = 1$

$t = 1 \; in \; F(2t+1, 3t-4, t+3)$

$= (2 \times 1 + 1, 3(1) - 4, 1 + 3) = (3, -1, 4)$

Therefore, the coordinate of F is $(3, -1, 4)$

$\overrightarrow{DF} = (2t+2)\hat{\imath} + (3t-6)\hat{\jmath} + t\hat{k}$

where $t = 1$

$\overrightarrow{DF} = (2 \times 1 + 2)\hat{\imath} + (3 \times 1 - 6)\hat{\jmath} + 1\hat{k}$

$\overrightarrow{DF} = 4\hat{\imath} - 3\hat{\jmath} + \hat{k}$

Now, the perpendicular distance from the given point to the given line is

$DF = \left|\overrightarrow{DF}\right|$

$= \sqrt{4^2 + (-3)^2 + 1^2} = \sqrt{16 + 9 + 1}$

$= \sqrt{26}$ *units*

1. *Find the parametric form of vector equation and Cartesian equation of a straight line passing through* $(5, 2, 8)$ *and is a perpendicular to the straight line* $\vec{r} = (\hat{\imath} + \hat{\jmath} - \hat{k}) + s(2\hat{\imath} - 2\hat{\jmath} + \hat{k})$ *and* $\vec{r} = (2\hat{\imath} - \hat{\jmath} - 3\hat{k}) + t(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$

Given : Point: $(5, 2, 8) \Rightarrow \vec{a} = 5\hat{\imath} + 2\hat{\jmath} + 8\hat{k}$

Lines: $\vec{r} = (\hat{\imath} + \hat{\jmath} - \hat{k}) + s(2\hat{\imath} - 2\hat{\jmath} + \hat{k}) \Rightarrow \boxed{\vec{b} = 2\hat{\imath} - 2\hat{\jmath} + \hat{k}}$

$\vec{r} = (2\hat{\imath} - \hat{\jmath} - 3\hat{k}) + t(\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) \Rightarrow \boxed{\vec{d} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}}$

Required line is perpendicular to both $\vec{b}$ *and* $\vec{d}$

$$\therefore \; \vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = \hat{\imath}\overset{-6}{(-4-2)} - \hat{\jmath}\overset{+3}{(4-1)} + \hat{k}\overset{+6}{(4+2)}$$

$= -6\hat{\imath} - 3\hat{\jmath} + 6\hat{k}$

$\vec{b} \times \vec{d} = -3(2\hat{\imath} + \hat{\jmath} - 2\hat{k})$

$\therefore$ *The equation of the required line is* $\vec{r} = \vec{a} + t(\vec{b} \times \vec{d}), m \in R$

$\vec{r} = 5\hat{\imath} + 2\hat{\jmath} + 8\hat{k} + t(2\hat{\imath} + \hat{\jmath} - 2\hat{k})$

Cartesian equation of the straight line passing through $\underset{x_1, y_1, z_1}{(5, 2, 8)}$ *and parallel to the straight lines* $\vec{b} \times \vec{d}$

$\vec{b} \times \vec{d} = -3(2\hat{\imath} + \hat{\jmath} - 2\hat{k})$

$\vec{b}_1 = 2, \vec{b}_2 = 1, \vec{b}_3 = -2$

$$\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{b_3} \Rightarrow \frac{x-5}{2} = \frac{y-2}{1} = \frac{z-8}{-2}$$

2. *Show that the lines* $\vec{r} = (6\hat{\imath} + \hat{\jmath} + 2\hat{k}) + s(\hat{\imath} + 2\hat{\jmath} - 3\hat{k})$ **and** $\vec{r} = (3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}) + t(2\hat{\imath} + 4\hat{\jmath} - 5\hat{k})$ ***are skew lines and hence find the shortest distance between them.***

Given: Lines: $\vec{r} = \underbrace{(6\hat{\imath} + \hat{\jmath} + 2\hat{k})}_{\vec{a}} + s\underbrace{(\hat{\imath} + 2\hat{\jmath} - 3\hat{k})}_{\vec{b}}$

and $\vec{r} = \underbrace{(3\hat{\imath} + 2\hat{\jmath} - 2\hat{k})}_{\vec{c}} + t\underbrace{(2\hat{\imath} + 4\hat{\jmath} - 5\hat{k})}_{\vec{d}}$

$\vec{a} = 6\hat{\imath} + \hat{\jmath} + 2\hat{k}, \vec{b} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$

$\vec{c} = 3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}, \vec{d} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$

Since $\vec{b} \neq \vec{d}$, *they are not parallel*

To find shortest distance:

Shortest distance between the two skew lines $\boldsymbol{\delta = \frac{|(\vec{c} - \vec{a}).(\vec{b} \times \vec{d})|}{|\vec{b} \times \vec{d}|}}$

$$\therefore \vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = \hat{\imath}[\overset{+2}{(-10 + 12)}] - \hat{\jmath}[\overset{+1}{(-5 + 6)}] + \hat{k}[\overset{0}{(4 - 4)}]$$

$\vec{b} \times \vec{d} = 2\hat{\imath} - \hat{\jmath}$

$|\vec{b} \times \vec{d}| = \sqrt{2^2 + (-1)^2} = \sqrt{4 + 1}$

$|\vec{b} \times \vec{d}| = \sqrt{\mathbf{5}}$

$\vec{c} - \vec{a} = (3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}) - (6\hat{\imath} + \hat{\jmath} + 2\hat{k}) = 3\hat{\imath} + 2\hat{\jmath} - 2\hat{k} - 6\hat{\imath} - \hat{\jmath} - 2\hat{k}$

$\vec{c} - \vec{a} = -3\hat{\imath} + \hat{\jmath} - 4\hat{k}$

$(\vec{c} - \vec{a}).(\vec{b} \times \vec{d}) = (-3\hat{\imath} + \hat{\jmath} - 4\hat{k}).(2\hat{\imath} - \hat{\jmath}) = -6 - 1$

$(\vec{c} - \vec{a}).(\vec{b} \times \vec{d}) = \mathbf{-7}$

$$\therefore \delta = \frac{|(\vec{c} - \vec{a}).(\vec{b} \times \vec{d})|}{|\vec{b} \times \vec{d}|} = \frac{|-7|}{\sqrt{5}}$$

$$\boxed{\delta = \frac{7}{\sqrt{5}} \text{ units}}$$

3. If the two lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-m}{2}=z$ intersect at a point and find the value of m.

Given:

Line 1: $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ Line 2: $\frac{x-3}{1}=\frac{y-m}{2}=z$

Take: $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}=t \implies \frac{x-1}{2}=t, \frac{y+1}{3}=t, \frac{z-1}{4}=t$

$x-1=2t,\ y+1=3t, z-1=4t \implies x=2t+1, y=3t-1, z=4t+1$

Any point on this line is of the form $(2t+1, 3t-1, 4t+1)$

Take: $\frac{x-3}{1}=\frac{y-m}{2}=\frac{z-0}{1}=s \implies \frac{x-3}{1}=s, \frac{y-m}{3}=s, \frac{z-0}{1}=s$

$x-3=s, y-m=2s, z-0=s \implies x=s+3,\ y=2s+m,\ z=s$

Any point on this line is of the form $(s+3, 2s+m, s)$

Since the lines are intersecting, $(2t+1, 3t-1, 4t+1)=(s+3, 2s+m, s)$

$2t+1=s+3 \implies 2t-s=3-1 \implies 2t-s=2 \ldots(1)$

$4t+1=s \implies 4t-s=-1 \ldots(2)$

Solve (1) and (2)

$(1) \implies 2t-s=2$

$(2) \implies \underset{(-)}{4t}\ \underset{(+)}{-\ s} = \underset{(+)}{-1}$

$-2t=3 \implies t=-\frac{3}{2}$

sub $t=-\frac{3}{2}$ in (1) $2t-s=2$

$2\left(\frac{-3}{2}\right)-s=2 \implies -3-s=2 \implies -s=2+3$

$-s=5 \implies s=-5$

To find the value of m. $3t-1=2s+m$

Sub $t=-\frac{3}{2}$ and $s=-5$

$3\left(\frac{-3}{2}\right)-1=2(-5)+m \implies \frac{-9}{2}-1=-10+m$

$\frac{-9}{2}-1+10=m \implies \frac{-9}{2}+9=m \implies m=\frac{-9+18}{2}$

$$\boxed{m=\frac{9}{2}}$$

4. *If the two lines* $\mathbf{\frac{x-3}{3}=\frac{y-3}{-1}, z-1=0}$ **and** $\mathbf{\frac{x-6}{2}=\frac{z-1}{3}, y-2=0}$ ***intersect. Find the point of intersection.***

Line 1: $\frac{x-3}{3}=\frac{y-3}{-1}, z-1=0$ *Line 2:* $\frac{x-6}{2}=\frac{z-1}{3}=y-2$

Take: $\frac{x-3}{3}=\frac{y-3}{-1}, z=1$ *and* *Take:* $\frac{x-6}{2}=\frac{z-1}{3}, y=2$

Since the lines are intersecting,

$$sub\ y=2\ in \frac{x-3}{3}=\frac{y-3}{-1}$$

$$\frac{x-3}{3}=\frac{2-3}{-1} \Rightarrow \frac{x-3}{3}=\frac{-1}{-1} \Rightarrow \frac{x-3}{3}=1 \Rightarrow x=6$$

The point of intersection is $(6,2,1)$

5. *Show that the straight lines* $\mathbf{x+1=2y=-12z}$ ***and*** $\mathbf{x=y+2=6z-6}$ ***are skew and hence find the shortest distance between them.***

Given: *Line 1:* $x+1=2y=-12z$ *Line 2:* $x=y+2=6z-6$

$$\frac{x-(-1)}{1}=\frac{y-0}{1/2}=\frac{z-0}{-1/12} \text{ and } \frac{x-0}{1}=\frac{y-(-2)}{1}=\frac{z-1}{1/6}$$

$$\therefore \vec{a}=-\hat{\imath} \text{ and } \vec{b}=\hat{\imath}+\frac{1}{2}\hat{\jmath}-\frac{1}{12}\hat{k} \quad \therefore \vec{c}=-2\hat{\jmath}+\hat{k}, \hat{d}=\hat{\imath}+\hat{\jmath}+\frac{1}{6}\hat{k}$$

Since $\vec{b}\neq\vec{d}$*, they are not parallel*

To find shortest distance:

Shortest distance between the two skew lines $\delta=\frac{|(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}$

$$\vec{c}-\vec{a}=-2\hat{\jmath}+\hat{k}-(-\hat{\imath})$$

$$\vec{c}-\vec{a}=\hat{\imath}-2\hat{\jmath}+\hat{k}$$

$$\therefore \vec{b}\times\vec{d}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & 1/2 & -1/12\\ 1 & 1 & 1/6\end{vmatrix}=\hat{\imath}\left(\frac{1}{12}+\frac{1}{12}\right)-\hat{\jmath}\left(\frac{1}{6}+\frac{1}{12}\right)+\hat{k}\left(1-\frac{1}{2}\right)$$

$$=\hat{\imath}\left(\frac{1}{12}+\frac{1}{12}\right)-\hat{\jmath}\left(\frac{1}{6}+\frac{1}{12}\right)+\hat{k}\left(1-\frac{1}{2}\right)=\hat{\imath}\left(\frac{2}{12}\right)-\hat{\jmath}\left(\frac{2+1}{12}\right)+\hat{k}\left(\frac{1}{2}\right)$$

$$\vec{b}\times\vec{d}=\hat{\imath}\left(\frac{\cancel{2}^{\,1}}{\cancel{12}_{\,6}}\right)-\hat{\jmath}\left(\frac{\cancel{3}^{\,1}}{\cancel{12}_{\,4}}\right)+\hat{k}\left(\frac{1}{2}\right) \Rightarrow \vec{b}\times\vec{d}=\frac{1}{6}\hat{\imath}-\frac{1}{4}\hat{\jmath}+\frac{1}{2}\hat{k}$$

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d}) = (\hat{\imath}-2\hat{\jmath}+\hat{k}).\left(\frac{1}{6}\hat{\imath}-\frac{1}{4}\hat{\jmath}+\frac{1}{2}\hat{k}\right)$$

$$=\frac{1}{6}+\frac{2}{4}+\frac{1}{2}=\frac{1}{6}+\frac{1}{2}+\frac{1}{2}=\frac{1}{6}+1=\frac{1+6}{6}$$

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=\frac{7}{6}$$

$$\therefore |\vec{b}\times\vec{d}| = \sqrt{\left(\frac{1}{6}\right)^2+\left(\frac{1}{4}\right)^2+\left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{36}+\frac{1}{16}+\frac{1}{4}} = \sqrt{\frac{4+9+36}{144}}$$

$$|\vec{b}\times\vec{d}| = \sqrt{\frac{49}{144}}$$

$$|\vec{b}\times\vec{d}| = \frac{7}{12}$$

Shortest distance between the skew lines $\boldsymbol{\delta} = \dfrac{|(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}$

$$=\frac{7/6}{7/12}=\frac{7}{6}\times\frac{12}{7}$$

$$\delta = 2$$

Shortest distance between the skew lines $= 2$ *units*

6. *Find the parametric form of vector equation of the straight line passing through* $(-1,2,1)$ ***and parallel to the straight line*** $\vec{r}=(2\hat{\imath}+3\hat{\jmath}-\hat{k})+t(\hat{\imath}-2\hat{\jmath}+\hat{k})$ ***and hence find the shortest between the lines***

Given: $\vec{r}=(2\hat{\imath}+3\hat{\jmath}-\hat{k})+t(\hat{\imath}-2\hat{\jmath}+\hat{k})$

$\vec{a}=-\hat{\imath}+2\hat{\jmath}+\hat{k}$ *and parallel vector:* $\vec{b}=\hat{\imath}-2\hat{\jmath}+\hat{k}$

$\therefore$ *Equation is* $\vec{r}=\vec{a}+t\vec{b}$

$$\vec{r}=(-\hat{\imath}+2\hat{\jmath}+\hat{k})+t(\hat{\imath}-2\hat{\jmath}+\hat{k})$$

Given line: $\vec{r}=\underbrace{(2\hat{\imath}+3\hat{\jmath}-\hat{k})}_{\vec{c}}+t\underbrace{(\hat{\imath}-2\hat{\jmath}+\hat{k})}_{\vec{d}}$

$$\vec{c}-\vec{a}=(2\hat{\imath}+3\hat{\jmath}-\hat{k})-(-\hat{\imath}+2\hat{\jmath}+\hat{k})$$

$$=2\hat{\imath}+3\hat{\jmath}-\hat{k}+\hat{\imath}-2\hat{\jmath}-\hat{k}$$

$$\boxed{\vec{c}-\vec{a}=3\hat{\imath}+\hat{\jmath}-2\hat{k}}$$

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$$(\vec{c}-\vec{a})\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix} = \hat{\imath}(1-4) - \hat{\jmath}(3+2) + \hat{k}(-6-1)$$

$$(\vec{c}-\vec{a})\times\vec{b} = -3\hat{\imath} - 5\hat{\jmath} - 7\hat{k}$$

$$\left|(\vec{c}-\vec{a})\times\vec{b}\right| = \sqrt{(-3)^2+(-5)^2+(-7)^2} = \sqrt{9+25+49}$$

$$\left|(\vec{c}-\vec{a})\times\vec{b}\right| = \sqrt{83}$$

$$\left|\vec{b}\right| = \sqrt{1^2+(-2)^2+1^2} = \sqrt{1+4+1}$$

$$\left|\vec{b}\right| = \sqrt{6}$$

Shortest distance between the lines $\delta = \dfrac{\left|(\vec{c}-\vec{a})\times\vec{b}\right|}{\left|\vec{b}\right|} = \dfrac{\sqrt{83}}{\sqrt{6}}$

$$\delta = \sqrt{\frac{83}{6}}$$

Shortest distance between the lines $\sqrt{\dfrac{83}{6}}$ *units*

7. *Find the foot of the perpendicular drawn from the point* $(5,4,2)$ ***to the line*** $\dfrac{x+1}{2} = \dfrac{y-3}{3} = \dfrac{z-1}{-1}$. ***Also find the equation of the perpendicular***

Line: $\dfrac{x+1}{2} = \dfrac{y-3}{3} = \dfrac{z-1}{-1} \Rightarrow \vec{b} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$

$$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1} = t$$

$$\frac{x+1}{2} = t, \frac{y-3}{3} = t, \frac{z-1}{-1} = t$$

$$x+1 = 2t, y-3 = 3t, z-1 = -t$$

$$x = 2t-1,\ y = 3t+3,\ z = -t+1$$

point F on the given line is $(2t-1, 3t+3, -t+1) \ldots (1)$

$$\overrightarrow{OF} = (2t-1)\hat{\imath} + (3t+3)\hat{\jmath} + (-t+1)\hat{k}$$

Given point is $D(5,4,2)$

$$\overrightarrow{OD} = 5\hat{\imath} + 4\hat{\jmath} + 2\hat{k}$$

$$\therefore \overrightarrow{DF} = \overrightarrow{OF} - \overrightarrow{OD}$$

$$= (2t-1)\hat{\imath} + (3t+3)\hat{\jmath} + (-t+1)\hat{k} - 5\hat{\imath} - 4\hat{\jmath} - 2\hat{k}$$

$$\overrightarrow{DF} = (2t-1-5)\hat{\imath} + (3t+3-4)\hat{\jmath} + (-t+1-2)\hat{k}$$

$$\overrightarrow{DF} = (2t-6)\hat{\imath} + (3t-1)\hat{\jmath} + (-t-1)\hat{k}$$

Since $\vec{b} \perp \overrightarrow{DF}$

$\vec{b}.\overrightarrow{DF} = 0 \Rightarrow (2\hat{\imath} + 3\hat{\jmath} - \hat{k}).\left[(2t-6)\hat{\imath} + (3t-1)\hat{\jmath} + (-t-1)\hat{k}\right] = 0$

$$2(2t-6) + 3(3t-1) - 1(-t-1) = 0$$

$$4t - 12 + 9t - 3 + t + 1 = 0$$

$$14t - 14 = 0 \Rightarrow t - 1 = 0 \Rightarrow t = 1$$

$t = 1 \; in \; \overrightarrow{OF} = (2t-1)\hat{\imath} + (3t+3)\hat{\jmath} + (-t+1)\hat{k}$

$F \; is \; (2(1) - 1, 3(1) + 3, -1 + 1) = (1, 6, 0)$

$\therefore$ *Foot of the perpendicular is* $(\mathbf{1}, \mathbf{6}, \mathbf{0})$

$\therefore$ *Equation of the perpendicular DF is the equation of the line passing through two points* $(5, 4, 2)$ *and* $(1, 6, 0)$.

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \Rightarrow \frac{x-5}{1-5} = \frac{y-4}{6-4} = \frac{z-2}{0-2}$$

$$\frac{x-5}{-4} = \frac{y-4}{2} = \frac{z-2}{-2}$$

$\therefore$ *Required equation is* $\dfrac{x-5}{-4} = \dfrac{y-4}{2} = \dfrac{z-2}{-2}$

Excercise 6.6

***Example 6.38**: Find the vector and Cartesian form of the equations of a plane which is at a distance of 12 units from the origin and perpendicular to $6\hat{i}+2\hat{j}-3\hat{k}$.*

$$Let\ \vec{d}=6\hat{\imath}+2\hat{\jmath}-3\hat{k}, p=12$$

$$\hat{d}=\frac{\vec{d}}{|\vec{d}|}\Rightarrow \hat{d}=\frac{6\hat{\imath}+2\hat{\jmath}-3\hat{k}}{|6\hat{\imath}+2\hat{\jmath}-3\hat{k}|}\Rightarrow \hat{d}=\frac{6\hat{\imath}+2\hat{\jmath}-3\hat{k}}{\sqrt{6^2+2^2+(-3)^2}}$$

$$\hat{d}=\frac{6\hat{\imath}+2\hat{\jmath}-3\hat{k}}{\sqrt{36+4+9}}\Rightarrow \hat{d}=\frac{6\hat{\imath}+2\hat{\jmath}-3\hat{k}}{\sqrt{49}}\Rightarrow \hat{d}=\frac{1}{7}(6\hat{\imath}+2\hat{\jmath}-3\hat{k})$$

vector equation of the plane in normal form is $\vec{r}.\hat{d}=p$

$$\vec{r}.\frac{1}{7}(6\hat{\imath}+2\hat{\jmath}-3\hat{k})=12\Rightarrow \vec{r}.(6\hat{\imath}+2\hat{\jmath}-3\hat{k})=84$$

Cartesian equation of the required plane.

$$Let\ \vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$$

$$(x\hat{\imath}+y\hat{\jmath}+z\hat{k}).(6\hat{\imath}+2\hat{\jmath}-3\hat{k})=84$$

$$6x+2y-3z=84$$

***Example 6.39**: If the Cartesian equation of a plane is $3x-4y+3z=-8$, find the vector equation of the plane in the standard form.*

$$Let\ \vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$$

$$3x-4y+3z=-8$$

$$(x\hat{\imath}+y\hat{\jmath}+z\hat{k}).(3\hat{\imath}-4\hat{\jmath}+3\hat{k})=-8$$

$$\vec{r}.(3\hat{\imath}-4\hat{\jmath}+3\hat{k})=-8$$

which is the vector equation of the given plane in standard form.

***Example 6.40**: Find the direction cosines and length of the perpendicular from the origin to the plane $\vec{r}.(3\hat{i}-4\hat{j}+12\hat{k})=5$.*

$$\vec{r}.(3\hat{\imath}-4\hat{\jmath}+12\hat{k})=5$$

$$\vec{r}.\vec{d}=q$$

$$Let\ \vec{d}=3\hat{\imath}-4\hat{\jmath}+12\hat{k}, q=5$$

$$|\vec{d}|=\sqrt{3^2+(-4)^2+12^2}\Rightarrow |\vec{d}|=\sqrt{9+16+144}$$

$$|\vec{d}|=\sqrt{169}\Rightarrow |\vec{d}|=13$$

$$p=\frac{q}{|\vec{d}|}\Rightarrow p=\frac{5}{13}$$

$$\hat{d}=\frac{\vec{d}}{|\vec{d}|}\Rightarrow \hat{d}=\frac{3\hat{\imath}-4\hat{\jmath}+12\hat{k}}{13}$$

vector equation of the plane in normal form is $\vec{r}.\hat{d} = p$

$$\vec{r}.\left(\frac{3\hat{\imath} - 4\hat{\jmath} + 12\hat{k}}{13}\right) = \frac{5}{13}$$

$$\vec{r}.\left(\frac{3}{13}\hat{\imath} - \frac{4}{13}\hat{\jmath} + \frac{12}{13}\hat{k}\right) = \frac{5}{13}$$

The direction cosines of $\hat{d}$ are $\frac{3}{13}, -\frac{4}{13}, \frac{12}{13}$ and

The length of the perpendicular from the origin to the plane is $\frac{5}{13}$

Example 6.41: Find the vector and Cartesian equations of the plane passing through the point with position vector $4\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ and normal to vector $2\hat{\imath} - \hat{\jmath} + \hat{k}$.

Given : $\vec{a} = 4\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ and $\vec{n} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$,

Equation of the plane passing through a point and normal to a vector is

$$\vec{r}.\vec{n} = \vec{a}.\vec{n}$$

$$\vec{r}.(2\hat{\imath} - \hat{\jmath} + \hat{k}) = (4\hat{\imath} + 2\hat{\jmath} - 3\hat{k}).(2\hat{\imath} - \hat{\jmath} + \hat{k})$$

$$\vec{r}.(2\hat{\imath} - \hat{\jmath} + \hat{k}) = 8 - 2 - 3$$

$$\vec{r}.(2\hat{\imath} - \hat{\jmath} + \hat{k}) = 3$$

Cartesian equation of the plane

Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$(x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(2\hat{\imath} - \hat{\jmath} + \hat{k}) = 3$$

$$2x - y + z = 3$$

Example 6.42: A variable plane moves in such a way that the sum of the reciprocals of its intercepts on the coordinate axes is a constant. Show that the plane passes through a fixed point

The equation of the plane having intercepts a, b, c on the x, y, z axes respectively is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Since the sum of the reciprocals of the intercepts on the coordinate axes is a constant, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = k.$ where k is a constant

$\div k$

$$\frac{1}{a}\left(\frac{1}{k}\right) + \frac{1}{b}\left(\frac{1}{k}\right) + \frac{1}{c}\left(\frac{1}{k}\right) = 1.$$ Let us take $x = y = z = \frac{1}{k}$

This shows that the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and passes through the fixed point $\left(\frac{1}{k}, \frac{1}{k}, \frac{1}{k}\right)$

1. Find the parametric form of vector equation of a plane which is at a distance of 7 units from the origin having 3, −4, 5 as direction ratios of a normal to it.

Given $p = 7, \vec{d} = 3\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$

$$|\vec{d}| = \sqrt{3^2 + (-4)^2 + 5^2} \implies |\vec{d}| = \sqrt{9 + 16 + 25}$$

$$|\vec{d}| = \sqrt{50} \implies |\vec{d}| = \sqrt{25 \times 2} \implies |\vec{d}| = 5\sqrt{2}$$

$$\hat{d} = \frac{\vec{d}}{|\vec{d}|} \implies \hat{d} = \frac{3\hat{\imath} - 4\hat{\jmath} + 5\hat{k}}{5\sqrt{2}}$$

Equation of the plane is $\vec{r}.\hat{d} = p \implies r.\left(\frac{3\hat{\imath} - 4\hat{\jmath} + 5\hat{k}}{5\sqrt{2}}\right) = 7$

2. Find the direction cosines of the normal to the plane $12x + 3y - 4z = 65$. Also find the non − parametric form of vector equation of a plane and the length of the perpendicular to the plane from the origin

Given cartesian equation of the plane is $12x + 3y - 4z = 65$

vector equation of the given plane $\vec{r}.(12\hat{\imath} + 3\hat{\jmath} - 4\hat{k}) = 65$

$$\vec{d} = 12\hat{\imath} + 3\hat{\jmath} - 4\hat{k}, q = 65$$

$$|\vec{d}| = \sqrt{12^2 + 3^2 + (-4)^2} \implies |\vec{d}| = \sqrt{144 + 9 + 16}$$

$$|\vec{d}| = \sqrt{169} \implies |\vec{d}| = 13$$

$$p = \frac{q}{|\vec{d}|} \implies p = \frac{65}{13} = 5$$

The length of the perpendicular from the origin to the plane is 5

$$\hat{d} = \frac{\vec{d}}{|\vec{d}|} \implies \hat{d} = \frac{12\hat{\imath} + 3\hat{\jmath} - 4\hat{k}}{13}$$

Hence, the direction cosines is $\frac{12}{13}, \frac{3}{13}, \frac{-4}{13}$.

Non − parametric vector form of the equation of the plane is $\vec{r}.\hat{d} = p$

$$\vec{r}.\left(\frac{12\hat{\imath} + 3\hat{\jmath} - 4\hat{k}}{13}\right) = 5$$

3. Find the vector and cartesian equations of the plane passing through the point with position vector $2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}$ and normal to the vector $\hat{\imath} + 3\hat{\jmath} + 5\hat{k}$

Given: $\vec{a} = 2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}$, $\vec{n} = \hat{\imath} + 3\hat{\jmath} + 5\hat{k}$

Vector form of the equation of the plane passing through a point $(\vec{a})$ *and normal to a vector* $(\vec{n})$ *is* $\vec{r}.\vec{n} = \vec{a}.\vec{n}$

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$$\vec{r}.\vec{n} = \vec{a}.\vec{n}$$

$$\vec{r}.\left(\hat{\imath}+3\hat{\jmath}+5\hat{k}\right) = \left(2\hat{\imath}+6\hat{\jmath}+3\hat{k}\right).\left(\hat{\imath}+3\hat{\jmath}+5\hat{k}\right)$$

$$= 2+18+15$$

$$\vec{r}.\left(\hat{\imath}+3\hat{\jmath}+5\hat{k}\right) = 35$$

cartesian equation

$$Let\ \vec{r} = x\hat{\imath}+y\hat{\jmath}+z\hat{k}$$

$$\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right).\left(\hat{\imath}+3\hat{\jmath}+5\hat{k}\right) = 35$$

$$x+3y+5z = 35$$

4. *A plane passes through the point* $(1,1,2)$ *and the normal to the plane of magnitude* $3\sqrt{3}$ *makes equal acute angles with the coordinate axes. Find the equation of the plane.*

The plane passes through the point $\vec{a} = -\hat{\imath}+\hat{\jmath}+2\hat{k}$

Given: $|\vec{n}| = 3\sqrt{3}$

$\alpha = \beta = \gamma$ i.e $\cos\alpha = \cos\beta = \cos\gamma$ (*equal acute angle*)

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$$

$$cos^2\alpha+cos^2\alpha+cos^2\alpha = 1 \Longrightarrow 3cos^2\alpha = 1 \Longrightarrow cos^2\alpha = \frac{1}{3}$$

$$\cos\alpha = \frac{1}{\sqrt{3}}$$

$$\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{\sqrt{3}}$$

$$Direction\ cosines = (cos\alpha, cos\beta, cos\gamma) \Longrightarrow Direction\ cosines = \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

$\vec{n}$ = *magnitude and direction cosine*

$$\therefore \vec{n} = 3\sqrt{3}\left(\frac{1}{\sqrt{3}}\hat{\imath}+\frac{1}{\sqrt{3}}\hat{\jmath}+\frac{1}{\sqrt{3}}\hat{k}\right)$$

$$\vec{n} = 3\hat{\imath}+3\hat{\jmath}+3\hat{k}$$

$\therefore$ *The equation of required plane is* $\vec{r}.\vec{n} = \vec{a}.\vec{n}$

$$\vec{r}.\ (3\hat{\imath}+3\hat{\jmath}+3\hat{k}) = \left(-\hat{\imath}+\hat{\jmath}+2\hat{k}\right).\left(3\hat{\imath}+3\hat{\jmath}+3\hat{k}\right)$$

$$\vec{r}.\ (3\hat{\imath}+3\hat{\jmath}+3\hat{k}) = -3+3+6$$

$$\vec{r}.\ (3\hat{\imath}+3\hat{\jmath}+3\hat{k}) = 6 \Longrightarrow \vec{r}.3(\hat{\imath}+\hat{\jmath}+\hat{k}) = 6$$

$$\vec{r}.\left(\hat{\imath}+\hat{\jmath}+\hat{k}\right) = 2$$

cartesian equation of the required plane

$$Let\ \vec{r} = x\hat{\imath}+y\hat{\jmath}+z\hat{k}$$

$$\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right).\left(\hat{\imath}+\hat{\jmath}+\hat{k}\right) = 2$$

$$x+y+z = 2$$

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5. *Find the intercepts cut off by the plane $\vec{r}.(\widehat{6i}+4\hat{j}-3\hat{k})=12$ on the coordinate axes.*

$$\vec{r}.(\widehat{6i}+4\hat{j}-3\hat{k})=12$$

Let $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$

$$(x\hat{\imath}+y\hat{\jmath}+z\hat{k}).(\widehat{6i}+4\hat{j}-3\hat{k})=12$$

$$6x+4y-3z=12$$

$$\div 12$$

$$\frac{6x}{12}+\frac{4y}{12}-\frac{3z}{12}=1 \Rightarrow \frac{x}{2}+\frac{y}{3}-\frac{z}{4}=1$$

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{-4}=1$$

$\therefore$ *The* $x-$ *intercepts of the plane is* 2, y *intercept is* 3 *and* $z-$ *intercept is* -4

6. *If a plane meets the coordinate axes at A, B, C such that the centroid of the triangle ABC is the point (u,v,w) find the equation of the plane*

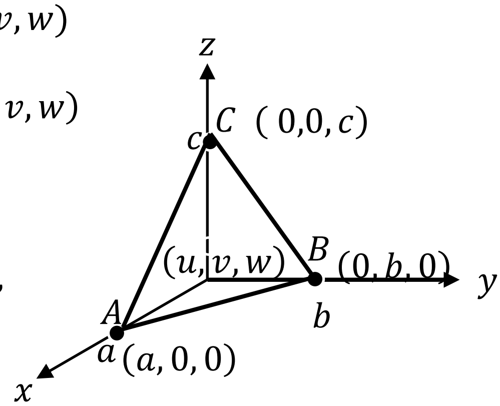
Let the intercepts of the plane with the coordinate axes be a, b, c *respectively.*

$\therefore$ *Equation of the plane in intercept form is* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Given that the centroid of $\Delta ABC = (u,v,w)$

$$\left(\frac{a+0+0}{3},\frac{0+b+0}{3},\frac{0+0+c}{3}\right)=(u,v,w)$$

$$\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)=(u,v,w)$$



Equating the like co − ordiantes we get,

$$u=\frac{a}{3},\ v=\frac{b}{3},\ w=\frac{c}{3}$$

$$a=3u, b=3v, c=3w$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{where } a=3u, b=3v, c=3w$$

$$\frac{x}{3u}+\frac{y}{3v}+\frac{z}{3w}=1 \Rightarrow \frac{1}{3}\left(\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right)=1$$

$$\frac{x}{u}+\frac{y}{v}+\frac{z}{w}=3 \text{ which is the required plane}$$

Excercise 6.7

Example 6.43: Find the non − parametric form of vector equation, and Cartesian equation of the plane passing through the point $(\mathbf{0},\mathbf{1},-\mathbf{5})$ ***and parallel to the straight lines*** $\vec{r}=(\hat{\imath}+2\hat{\jmath}-4\hat{k})+s(2\hat{\imath}+3\hat{\jmath}+6\hat{k})$ ***and*** $\vec{r}=(\hat{\imath}-3\hat{\jmath}+5\hat{k})+t(\hat{\imath}+\hat{\jmath}-\hat{k})$

Let $\vec{a}=\hat{\jmath}-5\hat{k}, \vec{b}=2\hat{\imath}+3\hat{\jmath}+6\hat{k}, \vec{c}=\hat{\imath}+\hat{\jmath}-\hat{k}$

$$(\vec{r}-\vec{a}).(\vec{b}\times\vec{c})=0$$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 3 & 6\\ 1 & 1 & -1\end{vmatrix}=\hat{\imath}(-3-6)-\hat{\jmath}(-2-6)+\hat{k}\,(2-3)$$

$$\vec{b}\times\vec{c}=-9\hat{\imath}+8\hat{\jmath}-\hat{k}$$

$$\left(\vec{r}-(\hat{\jmath}-5\hat{k})\right).(-9\hat{\imath}+8\hat{\jmath}-\hat{k})=0$$

$$[\vec{r}.(-9\hat{\imath}+8\hat{\jmath}-\hat{k})]-[(\hat{\jmath}-5\hat{k}).(-9\hat{\imath}+8\hat{\jmath}-\hat{k})]=0$$

$$[\vec{r}.(-9\hat{\imath}+8\hat{\jmath}-\hat{k})]-(8+5)=0$$

$$[\vec{r}.(-9\hat{\imath}+8\hat{\jmath}-\hat{k})]-13=0$$

$$\vec{r}.(-9\hat{\imath}+8\hat{\jmath}-\hat{k})=13$$

Cartesian equation: *Let* $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$

$$(x\vec{\imath}+y\vec{\jmath}+z\vec{k}).(-9\hat{\imath}+8\hat{\jmath}-\hat{k})=13$$

$$-9x+8y-z=13\Rightarrow\; 9x-8y+z=-13$$

$$9x-8y+z+13=0$$

Example 6.44: Find the vector parametric, vector non − parametric and Cartesian form of the equation of the plane passing through the points $(-\mathbf{1},\mathbf{2},\mathbf{0})$ ***and*** $(\mathbf{2},\mathbf{2},-\mathbf{1})$ ***and parallel to the straight line*** $\dfrac{x-1}{1}=\dfrac{2y+1}{2}=\dfrac{z+1}{-1}$

Vector equation : $\vec{r}=\vec{a}+s(\vec{b}-\vec{a})+t\vec{c}$

Here $\vec{a}=-\hat{\imath}+2\hat{\jmath}, \vec{b}=2\hat{\imath}+2\hat{\jmath}-\hat{k}$ *and* $\vec{c}=\hat{\imath}+\hat{\jmath}-\hat{k}$

$$\vec{b}-\vec{a}=2\hat{\imath}+2\hat{\jmath}-\hat{k}+\hat{\imath}-2\hat{\jmath}\Rightarrow\; \vec{b}-\vec{a}=3\hat{\imath}-\hat{k}$$

$$\vec{r}=-\hat{\imath}+2\hat{\jmath}+s(3\hat{\imath}-\hat{k})+t(\hat{\imath}+\hat{\jmath}-\hat{k})$$

Vector non – parametric equation : $(\vec{r}-\vec{a}).\left((\vec{b}-\vec{a})\times\vec{c}\right)=0$

$$(\vec{b}-\vec{a})\times\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 3 & 0 & -1\\ 1 & 1 & -1\end{vmatrix}=\hat{\imath}(0+1)-\hat{\jmath}(-3+1)+\hat{k}\,(3+0)$$

$$(\vec{b}-\vec{a})\times\vec{c}=\hat{\imath}+2\hat{\jmath}+3\hat{k}$$

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$$\left(\vec{r}-(-\hat{\imath}+2\hat{\jmath})\right).\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)=0$$

$$\left[\vec{r}.\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)\right]-\left[(-\hat{\imath}+2\hat{\jmath}).\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)\right]=0$$

$$\left[\vec{r}.\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)\right]-(-1+4)=0$$

$$\left[\vec{r}.\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)\right]-3=0$$

$$\vec{r}.\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)=3$$

Cartesian equation: *Let* $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$

$$\left(x\vec{\imath}+y\vec{\jmath}+z\vec{k}\right).\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)=3$$

$$x+2y+3z=3$$

$$x+2y+3z-3=0$$

1. *Find the non − parametric form of vector equation, and Cartesian of the equation of the plane passing through the point* $(2,3,6)$ *and parallel to the straight lines* $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-3}{1}$ *and* $\frac{x+3}{2}=\frac{y-3}{-5}=\frac{z+1}{-3}$

Vector non – parametric equation : $(\vec{r}-\vec{a}).\left(\vec{b}\times\vec{c}\right)=0$

Here $\vec{a}=2\hat{\imath}+3\hat{\jmath}+6\hat{k}$, $\vec{b}=2\hat{\imath}+3\hat{\jmath}+\hat{k}$ *and* $\vec{c}=2\hat{\imath}-5\hat{\jmath}-3\hat{k}$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 3 & 1\\ 2 & -5 & -3\end{vmatrix}=\hat{\imath}(-9+5)-\hat{\jmath}(-6-2)+\hat{k}\,(-10-6)$$

$$\vec{b}\times\vec{c}=-4\hat{\imath}+8\hat{\jmath}-16\hat{k}$$

$$\left(\vec{r}-\left(2\hat{\imath}+3\hat{\jmath}+6\hat{k}\right)\right).\left(-4\hat{\imath}+8\hat{\jmath}-16\hat{k}\right)=0$$

$$\left[\vec{r}.\left(-4\hat{\imath}+8\hat{\jmath}-16\hat{k}\right)\right]-\left[\left(2\hat{\imath}+3\hat{\jmath}+6\hat{k}\right).\left(-4\hat{\imath}+8\hat{\jmath}-16\hat{k}\right)\right]=0$$

$$\left[\vec{r}.\left(-4\hat{\imath}+8\hat{\jmath}-16\hat{k}\right)\right]-(-8+24-96)=0$$

$$\left[\vec{r}.\left(-4\hat{\imath}+8\hat{\jmath}-16\hat{k}\right)\right]+80=0$$

$$\vec{r}.\left(-4\hat{\imath}+8\hat{\jmath}-16\hat{k}\right)=-80$$

$$\div -4$$

$$\vec{r}.\left(\hat{\imath}-2\hat{\jmath}+4\hat{k}\right)=20$$

Cartesian equation: *Let* $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$

$$\left(x\vec{\imath}+y\vec{\jmath}+z\vec{k}\right).\left(\hat{\imath}-2\hat{\jmath}+4\hat{k}\right)=20$$

$$x-2y+4z-20=0$$

$$x-2y+4z=20$$

2. *Find the parametric form of vector equation, and Cartesian equations of the plane passing through the points* $(2,2,1),(9,3,6)$ *and perpendicular to the plane* $2x+6y+6z=9$

Vector equation : $\vec{r}=\vec{a}+s(\vec{b}-\vec{a})+t\vec{c}$

Here $\vec{a}=2\hat{\imath}+2\hat{\jmath}+\hat{k}$,$\vec{b}=9\hat{\imath}+3\hat{\jmath}+6\hat{k}$ *and* $\vec{c}=2\hat{\imath}+6\hat{\jmath}+6\hat{k}$

$\vec{b}-\vec{a}=9\hat{\imath}+3\hat{\jmath}+6\hat{k}-2\hat{\imath}-2\hat{\jmath}-\hat{k} \Rightarrow \vec{b}-\vec{a}=7\hat{\imath}+\hat{\jmath}+5\hat{k}$

$\vec{r}= 2\hat{\imath}+2\hat{\jmath}+\hat{k}+s(7\hat{\imath}+\hat{\jmath}+5\hat{k})+t(2\hat{\imath}+6\hat{\jmath}+6\hat{k})$

Cartesian equation :

$(x_1,y_1,z_1)=(2,2,1);(x_2,y_2,z_2)=(9,3,6);\ (c_1,c_2,c_3)=(2,6,6)$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} \overset{+}{x-2} & \overset{-}{y-2} & \overset{+}{z-1} \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix}=0$$

$(x-2)(6-30)-(y-2)(42-10)+(z-1)(42-2)=0$

$(x-2)(-24)-(y-2)(32)+(z-1)(40)=0$

$-24x+48-32y+64+40z-40=0$

$-24x-32y+40z+72=0$

$\div -8$

$3x+4y-5z-9=0$

$\therefore$ *The parametric form of vector equation is* $\vec{r}.(3\hat{\imath}+4\hat{\jmath}-5\hat{k})=9$

3. *Find parametric form of vector equation and Cartesian equations of the plane passing through the points* $(2,2,1),(1,-2,3)$ *and parallel to the straight line passing through the points* $(2,1,-3)$ *and* $(-1,5,-8)$.

Vector equation : $\vec{r}=\vec{a}+s(\vec{b}-\vec{a})+t\vec{c}$

Here $\vec{a}=2\hat{\imath}+2\hat{\jmath}+\hat{k}$, $\vec{b}=\hat{\imath}-2\hat{\jmath}+3\hat{k}$

$\vec{b}-\vec{a}=\hat{\imath}-2\hat{\jmath}+3\hat{k}-2\hat{\imath}-2\hat{\jmath}-\hat{k} \Rightarrow \vec{b}-\vec{a}=-\hat{\imath}-4\hat{\jmath}+2\hat{k}$

The straight line passing through the points $(\underset{x_1}{2},\underset{y_1}{1},\underset{z_1}{-3})$ *and* $(\underset{x_2}{-1},\underset{y_2}{5},\underset{z_2}{-8})$ *is*

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

$$\frac{x-2}{-1-2}=\frac{y-1}{5-1}=\frac{z+3}{-8+3} \Rightarrow \frac{x-2}{-3}=\frac{y-1}{4}=\frac{z+3}{-5}$$

$\vec{c}=-3\hat{\imath}+4\hat{\jmath}-5\hat{k}$

$\vec{r}= 2\hat{\imath}+2\hat{\jmath}+\hat{k}+s(-\hat{\imath}-4\hat{\jmath}+2\hat{k})+t(-3\hat{\imath}+4\hat{\jmath}-5\hat{k})$

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Cartesian equation :

$$(x_1, y_1, z_1) = (2,2,1); (x_2, y_2, z_2) = (1,-2,3);\ (c_1, c_2, c_3) = (-3,4,-5)$$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} \overset{+}{x-2} & \overset{-}{y-2} & \overset{+}{z-1} \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix} = 0$$

$$(x-2)(20-8) - (y-2)(5+6) + (z-1)(-4-12) = 0$$

$$(x-2)(12) - (y-2)(11) + (z-1)(-16) = 0$$

$$12x - 24 - 11y + 22 - 16z + 16 = 0$$

$$12x - 11y - 16z + 14 = 0$$

4. *Find the non − parametric form of vector equation of the plane passing through the point* $(1,-2,4)$ ***and perpendicular to the plane*** $x+2y-3z=11$ ***and parallel to the line*** $\frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$

Non – parametric Vector equation : $(\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$

Here $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 4\hat{k}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ *and* $\vec{c} = 3\hat{\imath} - \hat{\jmath} + \hat{k}$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix} = \hat{\imath}(2-3) - \hat{\jmath}(1+9) + \hat{k}(-1-6)$$

$$\vec{b} \times \vec{c} = -\hat{\imath} - 10\hat{\jmath} - 7\hat{k}$$

$$\left(\vec{r} - (\hat{\imath} - 2\hat{\jmath} + 4\hat{k})\right).(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k}) = 0$$

$$[\vec{r}.(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k})] - [(\hat{\imath} - 2\hat{\jmath} + 4\hat{k}).(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k})] = 0$$

$$[\vec{r}.(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k})] - (-1 + 20 - 28) = 0$$

$$[\vec{r}.(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k})] + 9 = 0$$

$$\vec{r}.(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k}) = -9$$

$$\times\ (-)$$

$$\vec{r}.(\hat{\imath} + 10\hat{\jmath} + 7\hat{k}) = 9$$

Cartesian equation: *Let* $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$(x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(\hat{\imath} + 10\hat{\jmath} + 7\hat{k}) = 9$$

$$x + 10y + 7z = 9$$

$$x + 10y + 7z - 9 = 0$$

5. *Find the parametric form of vector equation, and Cartesian equations of the plane containing the line* $\vec{r} = (\hat{\imath} - \hat{\jmath} + 3\hat{k}) + t(2\hat{\imath} - \hat{\jmath} + 4\hat{k})$ ***and perpendicular to plane*** $\vec{r}.(\hat{\imath} + 2\hat{\jmath} + \hat{k}) = 8$.

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Vector equation : $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$Here\ \vec{a} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$, $\vec{b} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$ and $\vec{c} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$

$$\vec{r} = \hat{\imath} - \hat{\jmath} + 3\hat{k} + s(2\hat{\imath} - \hat{\jmath} + 4\hat{k}) + t(\hat{\imath} + 2\hat{\jmath} + \hat{k})$$

Cartesian equation :

$(x_1, y_1, z_1) = (1, -1, 3); (b_1, b_2, b_3) = (2, -1, 4); (c_1, c_2, c_3) = (1, 2, 1)$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} \overset{+}{x - 1} & \overset{-}{y + 1} & \overset{+}{z - 3} \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$$(x - 1)(-1 - 8) - (y + 1)(2 - 4) + (z - 3)(4 + 1) = 0$$

$$(x - 1)(-9) - (y + 1)(-2) + (z - 3)(5) = 0$$

$$-9x + 9 + 2y + 2 + 5z - 15 = 0$$

$$-9x + 2y + 5z - 4 = 0$$

$$9x - 2y - 5z + 4 = 0$$

6. *Find the parametric vector, non − parametric vector and Cartesian form of the equations of the plane passing through the points* (3, 6, −2), (−1, −2, 6), *and* (6, −4, −2)

Vector equation: $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a})$

$$\vec{a} = 3\hat{\imath} + 6\hat{\jmath} - 2\hat{k},\ \vec{b} = -\hat{\imath} - 2\hat{\jmath} + 6\hat{k}\ \ and\ \vec{c} = 6\hat{\imath} - 4\hat{\jmath} - 2\hat{k}$$

$$\vec{b} - \vec{a} = -\hat{\imath} - 2\hat{\jmath} + 6\hat{k} - 3\hat{\imath} - 6\hat{\jmath} + 2\hat{k}$$

$$\vec{b} - \vec{a} = -4\hat{\imath} - 8\hat{\jmath} + 8\hat{k}$$

$$\vec{c} - \vec{a} = 6\hat{\imath} - 4\hat{\jmath} - 2\hat{k} - 3\hat{\imath} - 6\hat{\jmath} + 2\hat{k}$$

$$\vec{c} - \vec{a} = 3\hat{\imath} - 10\hat{\jmath}$$

$$\vec{r} = 3\hat{\imath} + 6\hat{\jmath} - 2\hat{k} + s(-4\hat{\imath} - 8\hat{\jmath} + 8\hat{k}) + t(3\hat{\imath} - 10\hat{\jmath})$$

Non – parametric Vector equation: $(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$

$$(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -4 & -8 & 8 \\ 3 & -10 & 0 \end{vmatrix} = \hat{\imath}(0 + 80) - \hat{\jmath}(0 - 24) + \hat{k}\,(40 + 24)$$

$$(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = 80\hat{\imath} + 24\hat{\jmath} + 64\hat{k}$$

$$[\vec{r} - (3\hat{\imath} + 6\hat{\jmath} - 2\hat{k})].(80\hat{\imath} + 24\hat{\jmath} + 64\hat{k}) = 0$$

$$\vec{r}.(80\hat{\imath} + 24\hat{\jmath} + 64\hat{k}) - (3\hat{\imath} + 6\hat{\jmath} - 2\hat{k}).(80\hat{\imath} + 24\hat{\jmath} + 64\hat{k}) = 0$$

$$\vec{r}.(80\hat{\imath} + 24\hat{\jmath} + 64\hat{k}) - (240 + 144 - 128) = 0$$

$$\vec{r}.(80\hat{\imath} + 24\hat{\jmath} + 64\hat{k}) - 256 = 0 \Rightarrow \vec{r}.(80\hat{\imath} + 24\hat{\jmath} + 64\hat{k}) = 256$$

$\div 8$

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$$\vec{r}.\left(10\hat{\imath}+3\hat{\jmath}+8\hat{k}\right)=32$$

$$\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right).\left(10\hat{\imath}+3\hat{\jmath}+8\hat{k}\right)=32$$

$$10x+3y+8z=32$$

7. *Find the no − parametric form of vector equation, and Cartesian equations of the plane*
$\boldsymbol{\vec{r}=\left(6\hat{\imath}-\hat{\jmath}+\hat{k}\right)+s\left(-\hat{\imath}+2\hat{\jmath}+\hat{k}\right)+t\left(-5\hat{\imath}-4\hat{\jmath}-5\hat{k}\right)}$.

Non – parametric Vector equation : $(\vec{r}-\vec{a}).\left(\vec{b}\times\vec{c}\right)=0$

Here $\vec{a}=6\hat{\imath}-\hat{\jmath}+\hat{k}$, $\vec{b}=-\hat{\imath}+2\hat{\jmath}+\hat{k}$ *and* $\vec{c}=-5\hat{\imath}-4\hat{\jmath}-5\hat{k}$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ -1 & 2 & 1\\ -5 & -4 & -5\end{vmatrix}=\hat{\imath}(-10+4)-\hat{\jmath}(5+5)+\hat{k}\,(4+10)$$

$$\vec{b}\times\vec{c}=-6\hat{\imath}-10\hat{\jmath}+14\hat{k}$$

$$\left(\vec{r}-\left(6\hat{\imath}-\hat{\jmath}+\hat{k}\right)\right).\left(-6\hat{\imath}-10\hat{\jmath}+14\hat{k}\right)=0$$

$$\left[\vec{r}.\left(-6\hat{\imath}-10\hat{\jmath}+14\hat{k}\right)\right]-\left[\left(6\hat{\imath}-\hat{\jmath}+\hat{k}\right).\left(-6\hat{\imath}-10\hat{\jmath}+14\hat{k}\right)\right]=0$$

$$\left[\vec{r}.\left(-6\hat{\imath}-10\hat{\jmath}+14\hat{k}\right)\right]-\ (-36+10+14)=0$$

$$\left[\vec{r}.\left(-6\hat{\imath}-10\hat{\jmath}+14\hat{k}\right)\right]+12=0$$

$$\div\ (-2)$$

$$\left[\vec{r}.\left(3\hat{\imath}+5\hat{\jmath}-7\hat{k}\right)\right]-6=0$$

Cartesian equation: *Let* $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$

$$\left(x\vec{\imath}+y\vec{\jmath}+z\vec{k}\right).\left(3\hat{\imath}+5\hat{\jmath}-7\hat{k}\right)-6=0$$

$$3x+5y-7z-6=0$$

Excercise 6.8

Example 6.45: Verify whether the line $\frac{x-3}{-4}=\frac{y-4}{-7}=\frac{z+3}{12}$ ***lies in the plane*** $5x-y+z=8$.

$Let\ (x_1,y_1,z_1)=(3,4,-3)\quad (a,b,c)=(-4,-7,12)\quad (A,B,C)=(5,-1,1)$

Condition for a line to lie in a plane $aA+bB+cC=0$

$$aA+bB+cC=(-4)(5)+(-7)(-1)+(12)(1)$$

$$=-20+7+12=-1\neq 0$$

Hence, the given line does not lie in the plane.

Example 6.46: Show that the lines $\vec{r}=(-\hat{\imath}-3\hat{\jmath}-5\hat{k})+s(3\hat{\imath}+5\hat{\jmath}+7\hat{k})$ ***and*** $\vec{r}=(2\hat{\imath}+4\hat{\jmath}+6\hat{k})+t(\hat{\imath}+4\hat{\jmath}+7\hat{k})$ ***are coplanar. Also, find the non − parametric form of vector equation of the plane containing these lines.***

$\vec{r}=(-\hat{\imath}-3\hat{\jmath}-5\hat{k})+s(3\hat{\imath}+5\hat{\jmath}+7\hat{k})$ *and* $\vec{r}=(2\hat{\imath}+4\hat{\jmath}+6\hat{k})+t(\hat{\imath}+4\hat{\jmath}+7\hat{k})$

Let $\vec{a}=-\hat{\imath}-3\hat{\jmath}-5\hat{k},\ \vec{b}=3\hat{\imath}+5\hat{\jmath}+7\hat{k},\ \vec{c}=2\hat{\imath}+4\hat{\jmath}+6\hat{k},\ \vec{d}=\hat{\imath}+4\hat{\jmath}+7\hat{k}$

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=0$$

$$\vec{c}-\vec{a}=2\hat{\imath}+4\hat{\jmath}+6\hat{k}-(-\hat{\imath}-3\hat{\jmath}-5\hat{k})$$

$$=2\hat{\imath}+4\hat{\jmath}+6\hat{k}+\hat{\imath}+3\hat{\jmath}+5\hat{k}$$

$$\vec{c}-\vec{a}=3\hat{\imath}+7\hat{\jmath}+11\hat{k}$$

$$\vec{b}\times\vec{d}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 3 & 5 & 7\\ 1 & 4 & 7\end{vmatrix}=\hat{\imath}(35-28)-\hat{\jmath}(21-7)+\hat{k}\,(12-5)$$

$$\vec{b}\times\vec{d}=7\hat{\imath}-14\hat{\jmath}+7\hat{k}$$

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=(3\hat{\imath}+7\hat{\jmath}+11\hat{k}).(7\hat{\imath}-14\hat{\jmath}+7\hat{k})$$

$$=21-98+77=98-98$$

$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=0$ $\therefore$ *The two given lines are coplanar*

To find the non parametric form of vector equation of the plane containing the two given non parallel coplanar lines.

$$(\vec{r}-\vec{a}).(\vec{b}\times\vec{d})=0$$

$$\left(\vec{r}-(-\hat{\imath}-3\hat{\jmath}-5\hat{k})\right).(7\hat{\imath}-14\hat{\jmath}+7\hat{k})=0$$

$$[\vec{r}.(7\hat{\imath}-14\hat{\jmath}+7\hat{k})]-[(-\hat{\imath}-3\hat{\jmath}-5\hat{k}).(7\hat{\imath}-14\hat{\jmath}+7\hat{k})]=0$$

$$[\vec{r}.(7\hat{\imath}-14\hat{\jmath}+7\hat{k})]-(-7+42-35)=0$$

$\vec{r}.(7\hat{\imath} - 14\hat{\jmath} + 7\hat{k}) = 0 \Rightarrow [\vec{r}.(7\hat{\imath} - 14\hat{\jmath} + 7\hat{k})] - 0 = 0$

$$\boxed{\vec{r}.(\hat{\imath} - 2\hat{\jmath} + \hat{k}) = 0}$$

1. Show that the straight lines $\vec{r} = (5\hat{\imath} + 7\hat{\jmath} - 3\hat{k}) + s(4\hat{\imath} + 4\hat{\jmath} - 5\hat{k})$ and $\vec{r} = (8\hat{\imath} + 4\hat{\jmath} + 5\hat{k}) + t(7\hat{\imath} + \hat{\jmath} + 3\hat{k})$ are coplanar. Find the form of vector equation of the plane in which they lie.

$\vec{r} = (5\hat{\imath} + 7\hat{\jmath} - 3\hat{k}) + s(4\hat{\imath} + 4\hat{\jmath} - 5\hat{k})$ *and* $\vec{r} = (8\hat{\imath} + 4\hat{\jmath} + 5\hat{k}) + t(7\hat{\imath} + \hat{\jmath} + 3\hat{k})$

Let $\vec{a} = 5\hat{\imath} + 7\hat{\jmath} - 3\hat{k}, \vec{b} = 4\hat{\imath} + 4\hat{\jmath} - 5\hat{k}, \vec{c} = 8\hat{\imath} + 4\hat{\jmath} + 5\hat{k}, \vec{d} = 7\hat{\imath} + \hat{\jmath} + 3\hat{k}$

$$(\vec{c} - \vec{a}).(\vec{b} \times \vec{d}) = 0$$

$\vec{c} - \vec{a} = 8\hat{\imath} + 4\hat{\jmath} + 5\hat{k} - (5\hat{\imath} + 7\hat{\jmath} - 3\hat{k})$

$= 8\hat{\imath} + 4\hat{\jmath} + 5\hat{k} - 5\hat{\imath} - 7\hat{\jmath} + 3\hat{k}$

$\vec{c} - \vec{a} = 3\hat{\imath} - 3\hat{\jmath} + 8\hat{k}$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = \hat{\imath}(12 + 5) - \hat{\jmath}(12 + 35) + \hat{k}\,(4 - 28)$$

$\vec{b} \times \vec{d} = 17\hat{\imath} - 47\hat{\jmath} - 24\hat{k}$

$(\vec{c} - \vec{a}).(\vec{b} \times \vec{d}) = (3\hat{\imath} - 3\hat{\jmath} + 8\hat{k}).(17\hat{\imath} - 47\hat{\jmath} - 24\hat{k})$

$= 51 + 141 - 192$

$= 192 - 192 = 0$ *Therefore the two given lines are coplanar.*

To find the non parametric form of vector equation of the plane containing the two given coplanar lines.

$$(\vec{r} - \vec{a}).(\vec{b} \times \vec{d}) = 0$$

$$\left(\vec{r} - (5\hat{\imath} + 7\hat{\jmath} - 3\hat{k})\right).(17\hat{\imath} - 47\hat{\jmath} - 24\hat{k}) = 0$$

$$[\vec{r}.(17\hat{\imath} - 47\hat{\jmath} - 24\hat{k})] - [(5\hat{\imath} + 7\hat{\jmath} - 3\hat{k}).(17\hat{\imath} - 47\hat{\jmath} - 24\hat{k})] = 0$$

$$[\vec{r}.(17\hat{\imath} - 47\hat{\jmath} - 24\hat{k})] - (85 - 329 + 72) = 0$$

$$[\vec{r}.(17\hat{\imath} - 47\hat{\jmath} - 24\hat{k})] - 172 = 0$$

$$\vec{r}.(17\hat{\imath} - 47\hat{\jmath} - 24\hat{k}) = 172$$

2. Show that the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{3}$ and $\frac{x-1}{-3} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar. Also, find the plane containing these lines.

Let $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k},\quad \vec{b} = \hat{\imath} + \hat{\jmath} + 3\hat{k},\quad \vec{c} = \hat{\imath} + 4\hat{\jmath} + 5\hat{k},\quad \vec{d} = -3\hat{\imath} + 2\hat{\jmath} + \hat{k}$

The two given lines are co − planar $(\vec{c} - \vec{a}).(\vec{b} \times \vec{d}) = 0$

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$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-1 & y-4 & z-5 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = 0$$

$$(x-1)(1-6)-(y-4)(1+9)+(z-5)(2+3)=0$$

$$(x-1)(-5)-(y-4)(10)+(z-5)(5)=0$$

$$-5x+5-10y+40+5z-25=0$$

$$-5x-10y+5z+20=0$$

$$\div(-5)$$

$$x+2y-z-4=0$$

Which is the equation of the plane containing the given lines.

3. *If the straight lines* $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{m^2}$ ***and*** $\frac{x-3}{1}=\frac{y-2}{m^2}=\frac{z-1}{2}$ ***are coplanar, find the distinct real values of m.***

Let $\vec{a}=\hat{\imath}+2\hat{\jmath}+3\hat{k}$, $\vec{b}=\hat{\imath}+2\hat{\jmath}+m^2\hat{k}$, $\vec{c}=\hat{\imath}+2\hat{\jmath}+\hat{k}$, $\vec{d}=\hat{\imath}+m^2\hat{\jmath}+2\hat{k}$

The two given lines are co − planar

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=0$$

$$\vec{c}-\vec{a}=\hat{\imath}+2\hat{\jmath}+\hat{k}-(\hat{\imath}+2\hat{\jmath}+3\hat{k})$$

$$=\hat{\imath}+2\hat{\jmath}+\hat{k}-\hat{\imath}-2\hat{\jmath}-3\hat{k}$$

$$\vec{c}-\vec{a}=-\hat{\imath}+\hat{\jmath}+\hat{k}$$

$$\vec{b}\times\vec{d}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix}=\hat{\imath}(1-6)-\hat{\jmath}(1+9)+\hat{k}(2+3)$$

$$\vec{b}\times\vec{d}=-5\hat{\imath}-10\hat{\jmath}+5\hat{k}$$

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=(-\hat{\imath}+\hat{\jmath}+\hat{k}).(-5\hat{\imath}-10\hat{\jmath}+5\hat{k})$$

$$=5-10+5=10-10$$

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=0$$

Therefore the two given lines are coplanar.

Cartesian equation :

$$(x_1,y_1,z_1)=(1,4,5);(b_1,b_2,b_3)=(1,1,3);\ (d_1,d_2,d_3)=(-3,2,1)$$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-1 & y-4 & z-5 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = 0$$

$$(x-1)(1-6)-(y-4)(1+9)+(z-5)(2+3)=0$$
$$(x-1)(-5)-(y-4)(10)+(z-5)(5)=0$$
$$-5x+5-10y+40+5z-25=0$$
$$-5x-10y+5z+20=0$$
$$\div\ (-5)$$
$$x+2y-z-4=0$$

Which is the equation of the plane containing the given lines.

3. If the straight lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{m^2}$ and $\frac{x-3}{1}=\frac{y-2}{m^2}=\frac{z-1}{2}$ are coplanar, find the distinct real values of m.

Let $\vec{a}=\hat{\imath}+2\hat{\jmath}+3\hat{k}$, $\vec{b}=\hat{\imath}+2\hat{\jmath}+m^2\hat{k}$, $\vec{c}=\hat{\imath}+2\hat{\jmath}+\hat{k}$, $\vec{d}=\hat{\imath}+m^2\hat{\jmath}+2\hat{k}$

The two given lines are co − planar

$$(\vec{c}-\vec{a}).\left(\vec{b}\times\vec{d}\right)=0$$
$$\vec{c}-\vec{a}=\hat{\imath}+2\hat{\jmath}+\hat{k}-\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)$$
$$=\hat{\imath}+2\hat{\jmath}+\hat{k}-\hat{\imath}-2\hat{\jmath}-3\hat{k}$$
$$\vec{c}-\vec{a}=-2\hat{k}$$
$$\vec{b}\times\vec{d}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & 2 & m^2\\ 1 & m^2 & 2\end{vmatrix}=\hat{\imath}(4-m^4)-\hat{\jmath}(2-m^2)+\hat{k}\,(m^2-2)$$
$$(\vec{c}-\vec{a}).\left(\vec{b}\times\vec{d}\right)=0$$
$$\left(-2\hat{k}\right).\left[\hat{\imath}(4-m^4)-\hat{\jmath}(2-m^2)+\hat{k}\,(m^2-2)\right]=0$$
$$-2(m^2-2)=0\implies m^2-2=0$$
$$m^2=2\implies m=\pm\sqrt{2}$$

4. If the straight lines $\frac{x-1}{2}=\frac{y+1}{\lambda}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{\lambda}$ are coplanar, find λ and equations of the planes containing these two lines.

Let $\vec{a}=\hat{\imath}-\hat{\jmath}$, $\vec{b}=2\hat{\imath}+\lambda\hat{\jmath}+2\hat{k}$, $\vec{c}=-\hat{\imath}-\hat{\jmath}$, $\vec{d}=5\hat{\imath}+2\hat{\jmath}+\lambda\hat{k}$

The two given lines are co − planar

$$(\vec{c}-\vec{a}).\left(\vec{b}\times\vec{d}\right)=0$$
$$\vec{c}-\vec{a}=-\hat{\imath}-\hat{\jmath}-(\hat{\imath}-\hat{\jmath})$$
$$=-\hat{\imath}-\hat{\jmath}-\hat{\imath}+\hat{\jmath}$$
$$\boxed{\vec{c}-\vec{a}=-2\hat{\imath}}$$

$$\vec{b}\times\vec{d}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & \lambda & 2\\ 5 & 2 & \lambda\end{vmatrix}=\hat{\imath}(\lambda^2-4)-\hat{\jmath}(2\lambda-10)+\hat{k}\,(4-5\lambda)$$

$$(\vec{c}-\vec{a}).\left(\vec{b}\times\vec{d}\right)=0$$

$$(-2\hat{\imath}).\left[\hat{\imath}(\lambda^2-4)-\hat{\jmath}(2\lambda-10)+\hat{k}\,(4-5\lambda)\right]=0$$

$$-2(\lambda^2-4)=0\Rightarrow\ \lambda^2-4=0$$

$$\lambda^2=4\Rightarrow\ \lambda=\pm\sqrt{4}$$

$$\lambda=\pm 2$$

Cartesian equation :

$(x_1, y_1, z_1)=(1,-1,0)$; $(b_1, b_2, b_3)=(2,2,2)$; $(d_1, d_2, d_3)=(5,2,2)$

$$\begin{vmatrix}x-x_1 & y-y_1 & z-z_1\\ b_1 & b_2 & b_3\\ d_1 & d_2 & d_3\end{vmatrix}=0 \quad\Rightarrow\begin{vmatrix}\overset{+}{x-1} & \overset{-}{y+1} & \overset{+}{z}\\ 2 & 2 & 2\\ 5 & 2 & 2\end{vmatrix}=0$$

$$(x-1)(4-4)-(y+1)(4-10)+(z)(4-10)=0$$

$$(x-1)(0)-(y+1)(-6)+z\,(-6)=0$$

$$6(y+1)-6z=0\Rightarrow\ 6(y+1-z)=0$$

$$y-z+1=0\ for\ \lambda=2$$

$$y+z+1=0\ for\ \lambda=-2$$

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Excercise 6.9

Example 6.47: Find the acute angle between the planes $\vec{r}.\left(2\hat{\imath}+2\hat{\jmath}+2\hat{k}\right)=11$ ***and*** $4x-2y+2z=15$.

$\vec{r}.\left(2\hat{\imath}+2\hat{\jmath}+2\hat{k}\right)=11 \text{ and } 4x-2y+2z=15$

$\vec{n}_1=2\hat{\imath}+2\hat{\jmath}+2\hat{k} \text{ and } \vec{n}_2=4\hat{\imath}-2\hat{\jmath}+2\hat{k}$

$$\cos\theta=\frac{|\vec{n}_1.\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$$

$$\cos\theta=\frac{\left|\left(2\hat{\imath}+2\hat{\jmath}+2\hat{k}\right).\left(4\hat{\imath}-2\hat{\jmath}+2\hat{k}\right)\right|}{\sqrt{2^2+2^2+2^2}\sqrt{4^2+(-2)^2+2^2}}$$

$$\cos\theta=\frac{8-4+4}{\sqrt{4+4+4}\sqrt{16+4+4}} \Rightarrow \cos\theta=\frac{8}{\sqrt{12}\sqrt{24}}$$

$$\cos\theta=\frac{8}{2\sqrt{3}\times2\sqrt{6}} \Rightarrow \cos\theta=\frac{2}{\sqrt{18}} \Rightarrow \cos\theta=\frac{2}{3\sqrt{2}}$$

$$\cos\theta=\frac{\sqrt{2}\times\sqrt{2}}{3\sqrt{2}} \Rightarrow \cos\theta=\frac{\sqrt{2}}{3} \Rightarrow \theta=\cos^{-1}\frac{\sqrt{2}}{3}$$

Example 6.48: Find the angle between the straight line $\vec{r}=\left(2\hat{\imath}+3\hat{\jmath}+\hat{k}\right)+t\left(\hat{\imath}-\hat{\jmath}+\hat{k}\right)$ ***and the plane*** $2x-y+z=5$.

Let $\vec{b}=\hat{\imath}-\hat{\jmath}+\hat{k}$ *and* $\vec{n}=2\hat{\imath}-\hat{\jmath}+\hat{k}$

Angle between line and plane is $\sin\theta=\dfrac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}$

$$\sin\theta=\frac{\left(\hat{\imath}-\hat{\jmath}+\hat{k}\right).\left(2\hat{\imath}-\hat{\jmath}+\hat{k}\right)}{\sqrt{1^2+(-1)^2+1^2}\sqrt{2^2+(-1)^2+1^2}}$$

$$\sin\theta=\frac{2+1+1}{\sqrt{1+1+1}\sqrt{4+1+1}}$$

$$\sin\theta=\frac{4}{\sqrt{3}\sqrt{6}} \Rightarrow \sin\theta=\frac{4}{\sqrt{18}} \Rightarrow \sin\theta=\frac{4}{\sqrt{2\times3\times3}}$$

$$\sin\theta=\frac{4}{3\sqrt{2}} \Rightarrow \sin\theta=\frac{4\sqrt{2}}{3\sqrt{2}\times\sqrt{2}} \Rightarrow \sin\theta=\frac{4\sqrt{2}}{3\times2}$$

$$\sin\theta=\frac{2\sqrt{2}}{3} \Rightarrow \theta=\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

Example 6.49: Find the distance from the point $(2,5,-3)$ ***to the plan*** $\vec{r}.\left(6\hat{\imath}-3\hat{\jmath}+2\hat{k}\right)=5$

$$\vec{r}.\left(6\hat{\imath}-3\hat{\jmath}+2\hat{k}\right)=5 \quad where\ \vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$$

$$\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right).\left(6\hat{\imath}-3\hat{\jmath}+2\hat{k}\right)=5 \Rightarrow \ 6x-3y+2z=5$$

$$6x-3y+2z-5=0$$

$$The\ distance\ from\ the\ point\ to\ the\ plane=\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$

$$a=6, b=-3, c=2, d=-5$$

$$(x_1,y_1,z_1)=(2,5,-3)$$

$$=\left|\frac{(6)(2)+(-3)(5)+(2)(-3)-5}{\sqrt{6^2+(-3)^2+2^2}}\right|=\left|\frac{12-15-6-5}{\sqrt{36+9+4}}\right|$$

$$=\left|\frac{12-26}{\sqrt{49}}\right|=\left|\frac{-14}{7}\right|\ =2\ units$$

Example 6.50: Find the distance of a point $(5,-5,-10)$ from the point of intersection of a straight line passing through the points $A(4,1,2)$ and $B(7,5,4)$ with the plane $x-y+z=5$.

The cartesian equation of the straight line joining A and B is
Here $(x_1, y_1, z_1)=(4,1,2)$ *and* $(x_2, y_2, z_2)=(7,5,4)$.

Cartesian equation:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \Rightarrow \ \frac{x-4}{3}=\frac{y-1}{4}=\frac{z-2}{2}$$

$$\frac{x-4}{3}=\frac{y-1}{4}=\frac{z-2}{2}=t \Rightarrow \ \frac{x-4}{3}=t, \frac{y-1}{4}=t, \frac{z-2}{2}=t$$

$$x=3t+4, y=4t+1, z=2t+2 \Rightarrow (3t+4, 4t+1, 2t+2)$$

$\therefore$ *An arbitrary point on the straight line is of the form* $(3t+4, 4t+1, 2t+2)$

To find the point of intersection of the straight line and the plane

$(3t+4, 4t+1, 2t+2)$ *in* $x-y+z=5$

$$3t+4-(4t+1)+2t+2=5$$

$$3t+4-4t-1+2t+2=5$$

$$t+5-5=0 \Rightarrow \ t=0$$

sub $t=0$ *in* $(3t+4, 4t+1, 2t+2) \Rightarrow (3\times 0+4, 4\times 0+1, 2\times 0+2)$

$\therefore$ *The point of intersection of the straight line is* $(4,1,2)$.

The distance between the two points $(\underset{x_1}{4}, \underset{y_1}{1}, \underset{z_1}{2})$ *and* $(\underset{x_2}{5}, \underset{y_2}{-5}, \underset{z_2}{-10})$ *is*

$$=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$$= \sqrt{(5-4)^2 + (-5-1)^2 + (-10-2)^2}$$

$$= \sqrt{(1)^2 + (-6)^2 + (-12)^2} = \sqrt{1 + 36 + 144}$$

$$= \sqrt{181}\ units$$

Example 6.51: ***Find the distance between the parallel planes*** $\boldsymbol{x + 2y - 2z + 1 = 0}$ ***and*** $\boldsymbol{2x + 4y - 4z + 5 = 0}$.

The distance between the parallel planes $x + 2y - 2z + 1 = 0$ *and*

$$2x + 4y - 4z + 5 = 0$$
$$\div 2$$
$$x + 2y - 2z + \frac{5}{2} = 0$$

$$a = 1,\ b = 2,\ c = -2,\ d_1 = 1,\ d_2 = \frac{5}{2}$$

$$Distance = \left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\frac{1 - \frac{5}{2}}{\sqrt{1^2 + 2^2 + (-2)^2}}\right|$$

$$= \left|\frac{\frac{2-5}{2}}{\sqrt{1+4+4}}\right| = \left|\frac{\frac{3}{2}}{\sqrt{9}}\right| = \left|\frac{\frac{3}{2}}{3}\right| = \left|\frac{3}{2} \times \frac{1}{3}\right|$$

$$Distance = \frac{1}{2}$$

Example 6.52: ***Find the distance between the planes*** $\boldsymbol{\vec{r}.(2\hat{i} - \hat{j} - 2\hat{k}) = 6}$ ***and*** $\boldsymbol{\vec{r}.(6\hat{i} - 3\hat{j} - 6\hat{k}) = 27}$.

$\vec{r}.(2\hat{i} - \hat{j} - 2\hat{k}) = 6$ *where* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$(x\hat{i} + y\hat{j} + z\hat{k}).(2\hat{i} - \hat{j} - 2\hat{k}) = 6 \Longrightarrow 2x - y - 2z = 6$

$\vec{r}.(6\hat{i} - 3\hat{j} - 6\hat{k}) = 27$ *where* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$(x\hat{i} + y\hat{j} + z\hat{k}).(6\hat{i} - 3\hat{j} - 6\hat{k}) = 27 \Longrightarrow 6x - 3y - 6z = 27$

$\div 3$

$2x - y - 2z = 9$

The Cartesian equation of the planes are $2x - y - 2z = 3$ *and* $2x - y - 2z = 9$

$$a = 2,\ b = -1,\ c = -2,\ d_1 = 6,\ d_2 = 9$$

$$Distance = \left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\frac{6 - 9}{\sqrt{2^2 + (-1)^2 + (-2)^2}}\right|$$

$$= \left|\frac{-3}{\sqrt{4+1+4}}\right| = \left|\frac{-3}{\sqrt{9}}\right| = \left|\frac{-3}{3}\right| = 1$$

Example 6.53: ***Find the equation of the plane passing through the intersection of the planes*** $\boldsymbol{\vec{r}.(\hat{i} + \hat{j} + \hat{k}) + 1 = 0}$ ***and*** $\boldsymbol{\vec{r}.(2\hat{i} - 3\hat{j} + 5\hat{k}) = 2}$ ***and the point*** $\boldsymbol{(-1, 2, 1)}$.

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$$\vec{r}.\left(\hat{\imath}+\hat{\jmath}+\hat{k}\right)+1=0 \text{ and } \vec{r}.\left(2\hat{\imath}-3\hat{\jmath}+5\hat{k}\right)-2=0$$

vector equation of a plane passing through the line of intersection of the planes $(\vec{r}.\vec{n}_1-d_1)+\lambda(\vec{r}.\vec{n}_2-d_2)=0$

$$\vec{r}.\left(\hat{\imath}+\hat{\jmath}+\hat{k}\right)+1+\lambda\left[\vec{r}.\left(2\hat{\imath}-3\hat{\jmath}+5\hat{k}\right)-2\right]=0$$

$$\left[\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right).\left(\hat{\imath}+\hat{\jmath}+\hat{k}\right)+1\right]+\lambda\left[\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right).\left(2\hat{\imath}-3\hat{\jmath}+5\hat{k}\right)-2\right]=0$$

$$[x+y+z+1]+\lambda(2x-3y+5z-2)=0$$

Since this plane passes through the point $(-1,2,1)$

$$[-1+2+1+1]+\lambda(2(-1)-3(2)+5(1)-2)=0$$

$$3+\lambda(-2-6+5-2)=0 \Longrightarrow 3+\lambda(-5)=0$$

$$-5\lambda=-3 \Longrightarrow \lambda=\frac{3}{5}$$

$$[x+y+z+1]+\lambda(2x-3y+5z-2)=0 \text{ where } \lambda=\frac{3}{5}$$

$$[x+y+z+1]+\frac{3}{5}(2x-3y+5z-2)=0$$

$$[x+y+z+1]+\frac{3}{5}(2x-3y+5z-2)=0$$

multiply by 5 *on both sides*

$$5(x+y+z+1)+3(2x-3y+5z-2)=0$$

$$5x+5y+5z+5+6x-9y+15z-6=0 \Longrightarrow 11x-4y+20z-1=0$$

Hence the required equation of the plane is $11x-4y+20z=1$.

***Example 6.54*: *Find the equation of the plane passing through the intersection of the planes* $2x+3y-z+7=0$ *and* $x+y-2z+5=0$ *and is perpendicular to the plane* $x+y-3z-5=0$.**

The equation of the plane passing through the intersection of the planes $2x+3y-z+7=0$ *and* $x+y-2z+5=0$ *is*

$$(2x+3y-z+7)+\lambda\,(x+y-2z+5)=0$$

$$2x+3y-z+7+\lambda\,x+\lambda y-2\lambda z+5\lambda=0$$

$$(2+\lambda)x+(3+\lambda)y+(-1-2\lambda)z+7+5\lambda=0$$

since this plane is perpendicular to the given plane $x+y-3z-5=0$

$$(1)(2+\lambda)+(1)(3+\lambda)+(-3)(-1-2\lambda)=0$$

$$2+\lambda+3+\lambda+3+6\lambda=0$$

$$2+3+3+8\lambda=0 \Longrightarrow 8+8\lambda=0 \Longrightarrow 8\lambda=-8$$

$$\lambda=-1$$

$$(2x+3y-z+7)+\lambda\,(x+y-2z+5)=0 \text{ where } \lambda=-1$$

$$(2x+3y-z+7)-(x+y-2z+5)=0$$

$$2x+3y-z+7-x-y+2z-5=0$$

$$x+2y+z+2=0$$

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***Example 6.55**: Find the image of the point whose position vector is $\hat{i}+2\hat{j}+3\hat{k}$ in the plane $\vec{r}.(\hat{i}+2\hat{j}+4\hat{k})=38$.*

Let $\vec{u}=\hat{\imath}+2\hat{\jmath}+3\hat{k}, \vec{n}=\hat{\imath}+2\hat{\jmath}+4\hat{k}, p=38$

Then the position vector of the image $\vec{v}$ of $\vec{v}=\vec{u}+\frac{2[p-(\vec{u}.\vec{n})]}{|\vec{n}|^2}\vec{n}$

$\boxed{|\vec{n}|^2=\vec{n}.\vec{n}}$

$$\vec{v}=(\hat{\imath}+2\hat{\jmath}+3\hat{k})+\frac{2\left[38-\left((\hat{\imath}+2\hat{\jmath}+3\hat{k}).(\hat{\imath}+2\hat{\jmath}+4\hat{k})\right)\right]}{(\hat{\imath}+2\hat{\jmath}+4\hat{k}).(\hat{\imath}+2\hat{\jmath}+4\hat{k})}(\hat{\imath}+2\hat{\jmath}+4\hat{k})$$

$$\vec{v}=(\hat{\imath}+2\hat{\jmath}+3\hat{k})+\frac{2\left(38-(1+4+12)\right)}{1+4+16}(\hat{\imath}+2\hat{\jmath}+4\hat{k})$$

$$\vec{v}=(\hat{\imath}+2\hat{\jmath}+3\hat{k})+\frac{2(38-17)}{21}(\hat{\imath}+2\hat{\jmath}+4\hat{k})$$

$$\vec{v}=(\hat{\imath}+2\hat{\jmath}+3\hat{k})+\left(\frac{2\times \cancel{21}}{\cancel{21}}\right)(\hat{\imath}+2\hat{\jmath}+4\hat{k})$$

$$\vec{v}=\hat{\imath}+2\hat{\jmath}+3\hat{k}+2\hat{\imath}+4\hat{\jmath}+8\hat{k} \Longrightarrow \vec{v}=3\hat{\imath}+6\hat{\jmath}+11\hat{k}$$

$\therefore$ *The image of the point with position vector* $\hat{\imath}+2\hat{\jmath}+3\hat{k}$ *is* $3\hat{\imath}+6\hat{\jmath}+11\hat{k}$

***Example 6.56**: Find the coordinates of the point where the straight line $\vec{r}=(2\hat{i}-\hat{j}+2\hat{k})+t(3\hat{i}+4\hat{j}+2\hat{k})$ intersects the plane $x-y+z-5=0$.*

The Cartesian equation of the line $\vec{r}=(2\hat{\imath}-\hat{\jmath}+2\hat{k})+t(3\hat{\imath}+4\hat{\jmath}+2\hat{k})$

$(x_1,y_1,z_1)=(2,-1,2)$ *and* $(b_1,b_2,b_3)=(3,4,2)$

$$\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3} \Longrightarrow \frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}$$

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}=t \Longrightarrow \frac{x-2}{3}=t, \frac{y+1}{4}=t, \frac{z-2}{2}=t$$

$x=3t+2,\ y=4t-1,\ z=2t+2$

$(3t+2, 4t-1, 2t+2)$

The point $(3t+2,4t-1,2t+2)$ lies on the plane $x-y+z-5=0$

$3t+2-(4t-1)+2t+2-5=0$

$3t+2-4t+1+2t+2-5=0$

$t+5-5=0 \Longrightarrow t=0$

sub $t=0$ *in* $(3t+2,4t-1,2t+2)$

$(3\times 0+2, 4\times 0-1, 2\times 0+2)=(2,-1,2)$

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$\therefore$ *The coordinates is* $(2,-1,2)$

1. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(2\hat{i}-7\hat{j}+4\hat{k})=3$ and $3x-5y+4z+11=0$, and the point $(-2,1,3)$.

$\vec{r}.(2\hat{i}-7\hat{j}+4\hat{k})=3$ *where* $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$(x\hat{i}+y\hat{j}+z\hat{k}).(2\hat{i}-7\hat{j}+4\hat{k})=3 \Rightarrow 2x-7y+4z=3$

Given equation of planes are $2x-7y+4z=3$ *and* $3x-5y+4z+11=0$

$$(2x-7y+4z-3)+\lambda(3x-5y+4z+11)=0 \ \dots(1)$$

Since the plane passes through the point $(-2,1,3)$

$$[2(-2)-7(1)+4(3)-3]+\lambda[3(-2)-5(1)+4(3)+11]=0$$

$$-4-7+12-3+\lambda[-6-5+12+11]=0$$

$$-2+12\lambda=0 \Rightarrow 12\lambda=2$$

$$\lambda=\frac{2}{12} \Rightarrow \lambda=\frac{1}{6}$$

$$Subs\ \lambda=\frac{1}{6}\ in\ (1)$$

$$(2x-7y+4z-3)+\frac{1}{6}(3x-5y+4z+11)=0$$

$\times 6$

$$6(2x-7y+4z-3)+(3x-5y+4z+11)=0$$

$$12x-42y+24z-18+3x-5y+4z+11=0$$

$15x-47y+28z-7=0$ *which is the required equation of the plane.*

2. Find the equation of the plane passing through the line of intersection of the planes $x+2y+3z=2$ and $x-y+z=3$, and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3,1,-1)$.

Given equation of planes are $x+2y+3z-2=0$ *and* $x-y+z-3=0$

To point of intersection of the planes

$$(x+2y+3z-2)+\lambda(x-y+z-3)=0 \dots(1)$$

$$x+2y+3z-2+\lambda x-\lambda y+\lambda z-3\lambda=0$$

$$x+\lambda x+2y-\lambda y+3z+\lambda z-2-3\lambda=0$$

$$(1+\lambda)x+(2-\lambda)y+(3+\lambda)z-2-3\lambda=0$$

The distance from $\underset{(x_1,\,y_1,\,z_1)}{(3,1,-1)}$ *to this plane is* $\frac{2}{\sqrt{3}}$

Here $a=(1+\lambda), b=(2-\lambda), c=(3+\lambda)$ *and* $d=-2-3\lambda$

The distance between the point and the plane is $\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$

$$\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right| = \frac{2}{\sqrt{3}}$$

$$\left|\frac{(1+\lambda)3+(2-\lambda)1+(3+\lambda)(-1)-2-3\lambda}{\sqrt{(1+\lambda)^2+(2-\lambda)^2+(3+\lambda)^2}}\right| = \frac{2}{\sqrt{3}}$$

$$\left|\frac{3+3\lambda+2-\lambda-3-\lambda-2-3\lambda}{\sqrt{1+\lambda^2+2\lambda+4+\lambda^2-4\lambda+9+\lambda^2+6\lambda}}\right| = \frac{2}{\sqrt{3}}$$

$$\left|\frac{-2\lambda}{\sqrt{3\lambda^2+4\lambda+14}}\right| = \frac{2}{\sqrt{3}} \Longrightarrow \frac{2\lambda}{\sqrt{3\lambda^2+4\lambda+14}} = \frac{2}{\sqrt{3}}$$

$$2\sqrt{3}\lambda = 2\sqrt{3\lambda^2+4\lambda+14} \Longrightarrow \sqrt{3}\lambda = \sqrt{3\lambda^2+4\lambda+14}$$

Squaring on both sides

$$3\lambda^2 = 3\lambda^2+4\lambda+14 \Longrightarrow 4\lambda+14=0$$

$$4\lambda = -14 \Longrightarrow \lambda = -\frac{7}{2}$$

Subs $\lambda = -\frac{7}{2}$ *in* (1) $(x+2y+3z-2)+\lambda(x-y+z-3)=0$

$$(x+2y+3z-2)-\frac{7}{2}(x-y+z-3)=0$$

$\times 2$

$$2(x+2y+3z-2)-7(x-y+z-3)=0$$

$$2x+4y+6z-4-7x+7y-7z+21=0$$

$$-5x+11y-z+17=0 \Longrightarrow 5x-11y+z-17=0$$

3. *Find the angle between the line* $\vec{r} = (2\hat{\imath}-\hat{\jmath}+\hat{k})+t(\hat{\imath}+2\hat{\jmath}-2\hat{k})$ ***and the plane*** $\vec{r}.(6\hat{\imath}+3\hat{\jmath}+2\hat{k})=8$.

Let $\vec{b} = \hat{\imath}+2\hat{\jmath}-2\hat{k}$ *and* $\vec{n} = 6\hat{\imath}+3\hat{\jmath}+2\hat{k}$

Angle between line and plane is

$$\sin\theta = \frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}$$

$$\sin\theta = \frac{(\hat{\imath}+2\hat{\jmath}-2\hat{k}).(6\hat{\imath}+3\hat{\jmath}+2\hat{k})}{\sqrt{1^2+2^2+(-2)^2}\sqrt{6^2+3^2+2^2}}$$

$$\sin\theta = \frac{6+6-4}{\sqrt{1+4+4}\sqrt{36+9+4}}$$

$$\sin\theta = \frac{8}{\sqrt{9}\sqrt{49}} \Rightarrow \sin\theta = \frac{8}{3\times 7} \Rightarrow \sin\theta = \frac{8}{21}$$

$$\theta = \sin^{-1}\left(\frac{8}{21}\right)$$

4. Find the angle between the planes $\vec{r}.(\hat{\imath}+\hat{\jmath}-2\hat{k}) = 3$ and $2x-2y+z=2$.

$$\vec{r}.(\hat{\imath}+\hat{\jmath}-2\hat{k}) = 3 \text{ and } 2x-2y+z=2$$

$$\vec{n}_1 = \hat{\imath}+\hat{\jmath}-2\hat{k} \text{ and } \vec{n}_2 = 2\hat{\imath}-2\hat{\jmath}+\hat{k}$$

$$\cos\theta = \frac{|\vec{n}_1.\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$$

$$\cos\theta = \frac{\left|(\hat{\imath}+\hat{\jmath}-2\hat{k}).(2\hat{\imath}-2\hat{\jmath}+\hat{k})\right|}{\sqrt{1^2+1^2+(-2)^2}\sqrt{2^2+(-2)^2+1^2}}$$

$$\cos\theta = \frac{|2-2-2|}{\sqrt{1+1+4}\sqrt{4+4+1}}$$

$$\cos\theta = \frac{|-2|}{\sqrt{6}\sqrt{9}} \Rightarrow \cos\theta = \frac{2}{3\sqrt{6}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)}$$

5. Find the equation of the plane which passes through the point $(3,4,-1)$ and is parallel to the plane $2x-3y+5z+7=0$. Also, find the distance between the twoplanes.

Equation of the given plane is $2x-3y+5z+7=0$

Equation of the plane parallel to the given plane is $2x-3y+5z+k=0$

Since this plane passes through the point $(3,4,-1)$.

$$2(3)-3(4)+5(-1)+k=0 \Rightarrow 6-12-5+k=0$$

$$-11+k=0 \Rightarrow k=11$$

$$k=11 \text{ in } 2x-3y+5z+k=0$$

Required plane is $2x-3y+5z+11=0$

Equation of the plane are $2x-3y+5z+7=0$ *and* $2x-3y+5z+11=0$

Distance between two parallel planes $d = \left|\frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right|$

$$a=2, b=-3, c=5, d_1=7, d_2=11$$

$$d = \left|\frac{7-11}{\sqrt{2^2+(-3)^2+5^2}}\right| = \left|\frac{-4}{\sqrt{4+9+25}}\right| = \left|\frac{-4}{\sqrt{38}}\right|$$

$$d = \frac{4}{\sqrt{38}}\ units$$

6. *Find the length of the perpendicular from the point* $(1,-2,3)$ *to the plane* $x-y+z=5$.

Length of perpendicular from (x_1,y_1,z_1) *to the plane* $ax+by+cz-d=0$

$$is\ \left|\frac{ax_1+by_1+cz_1-d}{\sqrt{a^2+b^2+c^2}}\right|$$

$\therefore$ *Length of perpendicular from* $\underset{(x_1,y_1,z_1)}{(1,-2,3)}$ *to the plane* $x-y+z-5=0$ *is*

$$a=1\ ,b=-1, c=1, d=-5$$

$$= \left|\frac{1(1)-1(-2)+1(3)-5}{\sqrt{1^2+(-1)^2+1^2}}\right| = \left|\frac{1+2+3-5}{\sqrt{1+1+1}}\right|$$

$$= \left|\frac{1}{\sqrt{3}}\right| = \frac{1}{\sqrt{3}}\ units$$

7. *Find the point of intersection of the line* $x-1=\frac{y}{2}=z+1$ *with the plane* $2x-y+2z=2$. *Also, find the angle between the line and the plane*.

Given equation of line is $x-1=\frac{y}{2}=z+1$

$$x-1=\frac{y}{2}=z+1=t \Longrightarrow\ x-1=t,\ \frac{y}{2}=t,\ z+1=t$$

$$x=t+1, y=2t,\ z=t-1$$

$\therefore$ *Any point on the line is of the form* $(t+1,2t,t-1)$

The point $(t+1,2t,t-1)$ *lies on the plane* $2x-y+2z=2$

$$2(t+1)-2t+2(t-1)=2 \Longrightarrow 2t+2-2t+2t-2=2$$

$$2t=2 \Longrightarrow\ t=1$$

$$t=1\ in (t+1,2t,t-1) \Longrightarrow\ (1+1,2(1),1-1)=(2,2,0)$$

The point of intersection of the plane and the line is $(2,2,0)$

$$Line\ x-1=\frac{y}{2}=z+1\ and\ plane\ 2x-y+2z=2$$

$$Let\ \vec{b}=\hat{\imath}+2\hat{\jmath}+\hat{k}\ and\ \vec{n}=2\hat{\imath}-\hat{\jmath}+2\hat{k}$$

$$Angle\ between\ line\ and\ plane\ is\ \sin\theta=\frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}$$

$$\sin\theta = \frac{(\hat{\imath}+2\hat{\jmath}+\hat{k}).(2\hat{\imath}-\hat{\jmath}+2\hat{k})}{\sqrt{1^2+2^2+1^2}\sqrt{2^2+(-1)^2+2^2}} \Rightarrow \sin\theta = \frac{2-2+2}{\sqrt{1+4+1}\sqrt{4+1+4}}$$

$$\sin\theta = \frac{2}{\sqrt{6}\sqrt{9}} \Rightarrow \sin\theta = \frac{2}{3\sqrt{6}} \Rightarrow \theta = \sin^{-1}\left(\frac{2}{3\sqrt{6}}\right)$$

8. *Find the coordinates of the foot of the perpendicular and length of the perpendicular from the point* $(4,3,2)$ *to the plane* $x+2y+3z=2$.

Given plane $x+2y+3z=2$.

vector form : $\vec{r}.(\hat{\imath}+2\hat{\jmath}+3\hat{k}) = 2$

Cartesian equations of the straight line through the point $(4,3,2)$ *and parallel to* $\hat{\imath}+2\hat{\jmath}+3\hat{k}$

$$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$$

Here $(x_1,y_1,z_1) = (4,3,2)$ & $(b_1,b_2,b_3) = (1,2,3)$

$$\frac{x-4}{1} = \frac{y-3}{2} = \frac{z-2}{3}$$

$$\frac{x-4}{1} = \frac{y-3}{2} = \frac{z-2}{3} = t \Rightarrow \frac{x-4}{1} = t, \frac{y-3}{2} = t, \frac{z-2}{3} = t$$

$$x = t+4, \quad y = 2t+3, \quad z = 3t+2 \Rightarrow (t+4, 2t+3, 3t+2)$$

The point $(t+4, 2t+3, 3t+2)$ *lies on the plane* $x+2y+3z=2$.

$$t+4+2(2t+3)+3(3t+2) = 2 \Rightarrow t+4+4t+6+9t+6 = 2$$

$$14t+16 = 2 \Rightarrow 14t = 2-16 \Rightarrow 14t = -14$$

$$t = \frac{-14}{14} \Rightarrow t = -1$$

$$t = -1 \text{ in } (t+4, 2t+3, 3t+2) \Rightarrow (-1+4, 2(-1)+3, 3(-1)+2)$$

$$= (3, -2+3, -3+2) = (3,1,-1)$$

The coordinates of the foot of the perpendicular is $(3,1,-1)$

$\therefore$ *Length of perpendicular from* $\underset{(x_1,y_1,z_1)}{(4,3,2)}$ *to the plane* $x+2y+3z-2=0$ *is*

$$a = 1, b = 2, c = 3, d = -2$$

$$= \left|\frac{ax_1+by_1+cz_1-d}{\sqrt{a^2+b^2+c^2}}\right| = \left|\frac{1(4)+2(3)+3(2)-2}{\sqrt{1^2+2^2+3^2}}\right| = \left|\frac{4+6+6-2}{\sqrt{1+4+9}}\right|$$

$$= \left|\frac{14}{\sqrt{14}}\right| = \frac{\sqrt{14}\times\sqrt{14}}{\sqrt{14}} = \sqrt{14} \text{ units}$$

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Excercise 6.7

Example 6.43: Find the non − parametric form of vector equation, and Cartesian equation of the plane passing through the point $(\mathbf{0},\mathbf{1},-\mathbf{5})$ ***and parallel to the straight lines*** $\vec{\boldsymbol{r}} = (\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}) + \boldsymbol{s}(2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}})$ ***and*** $\vec{\boldsymbol{r}} = (\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}) + \boldsymbol{t}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}})$

$Let\ \vec{a} = \hat{\jmath} - 5\hat{k}, \vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}, \vec{c} = \hat{\imath} + \hat{\jmath} - \hat{k}$

$$(\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix} = \hat{\imath}(-3-6) - \hat{\jmath}(-2-6) + \hat{k}\,(2-3)$$

$$\vec{b} \times \vec{c} = -9\hat{\imath} + 8\hat{\jmath} - \hat{k}$$

$$\left(\vec{r} - (\hat{\jmath} - 5\hat{k})\right).(-9\hat{\imath} + 8\hat{\jmath} - \hat{k}) = 0$$

$$[\vec{r}.(-9\hat{\imath} + 8\hat{\jmath} - \hat{k})] - [(\hat{\jmath} - 5\hat{k}).(-9\hat{\imath} + 8\hat{\jmath} - \hat{k})] = 0$$

$$[\vec{r}.(-9\hat{\imath} + 8\hat{\jmath} - \hat{k})] - (8+5) = 0$$

$$[\vec{r}.(-9\hat{\imath} + 8\hat{\jmath} - \hat{k})] - 13 = 0$$

$$\vec{r}.(-9\hat{\imath} + 8\hat{\jmath} - \hat{k}) = 13$$

$Cartesian\ equation$: $Let\ \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$(x\vec{\imath} + y\vec{\jmath} + z\vec{k}).(-9\hat{\imath} + 8\hat{\jmath} - \hat{k}) = 13$$

$$-9x + 8y - z = 13 \Rightarrow\ 9x - 8y + z = -13$$

$$9x - 8y + z + 13 = 0$$

Example 6.44: Find the vector parametric, vector non − parametric and Cartesian form of the equation of the plane passing through the points $(-\mathbf{1},\mathbf{2},\mathbf{0})$ ***and*** $(\mathbf{2},\mathbf{2},-\mathbf{1})$ ***and parallel to the straight line*** $\dfrac{\boldsymbol{x}-\mathbf{1}}{\mathbf{1}} = \dfrac{\mathbf{2}\boldsymbol{y}+\mathbf{1}}{\mathbf{2}} = \dfrac{\boldsymbol{z}+\mathbf{1}}{-\mathbf{1}}$

Vector equation : $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$

Here $\vec{a} = -\hat{\imath} + 2\hat{\jmath}$,$\vec{b} = 2\hat{\imath} + 2\hat{\jmath} - \hat{k}$ $and\ \vec{c} = \hat{\imath} + \hat{\jmath} - \hat{k}$

$$\vec{\mathrm{b}} - \vec{\mathrm{a}} = 2\hat{\imath} + 2\hat{\jmath} - \hat{k} + \hat{\imath} - 2\hat{\jmath} \Rightarrow \vec{\mathrm{b}} - \vec{\mathrm{a}} = 3\hat{\imath} - \hat{k}$$

$$\vec{r} = -\hat{\imath} + 2\hat{\jmath} + s(3\hat{\imath} - \hat{k}) + t(\hat{\imath} + \hat{\jmath} - \hat{k})$$

Vector non – parametric equation : $(\vec{r} - \vec{a}).\left((\vec{b} - \vec{a}) \times \vec{c}\right) = 0$

$$(\vec{b} - \vec{a}) \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \hat{\imath}(0+1) - \hat{\jmath}(-3+1) + \hat{k}\,(3+0)$$

$$(\vec{b} - \vec{a}) \times \vec{c} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$$

$$\left(\vec{r}-(-\hat{\imath}+2\hat{\jmath})\right).\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)=0$$

$$\left[\vec{r}.\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)\right]-\left[(-\hat{\imath}+2\hat{\jmath}).\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)\right]=0$$

$$\left[\vec{r}.\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)\right]-(-1+4)=0$$

$$\left[\vec{r}.\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)\right]-3=0$$

$$\vec{r}.\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)=3$$

Cartesian equation: *Let* $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$

$$\left(x\vec{\imath}+y\vec{\jmath}+z\vec{k}\right).\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)=3$$

$$x+2y+3z=3$$

$$x+2y+3z-3=0$$

1. *Find the non − parametric form of vector equation, and Cartesian of the equation of the plane passing through the point* $(2,3,6)$ *and parallel to the straight lines* $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-3}{1}$ *and* $\frac{x+3}{2}=\frac{y-3}{-5}=\frac{z+1}{-3}$

Vector non – parametric equation : $(\vec{r}-\vec{a}).\left(\vec{b}\times\vec{c}\right)=0$

Here $\vec{a}=2\hat{\imath}+3\hat{\jmath}+6\hat{k}$, $\vec{b}=2\hat{\imath}+3\hat{\jmath}+\hat{k}$ *and* $\vec{c}=2\hat{\imath}-5\hat{\jmath}-3\hat{k}$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 3 & 1\\ 2 & -5 & -3\end{vmatrix}=\hat{\imath}(-9+5)-\hat{\jmath}(-6-2)+\hat{k}\,(-10-6)$$

$$\vec{b}\times\vec{c}=-4\hat{\imath}+8\hat{\jmath}-16\hat{k}$$

$$\left(\vec{r}-\left(2\hat{\imath}+3\hat{\jmath}+6\hat{k}\right)\right).\left(-4\hat{\imath}+8\hat{\jmath}-16\hat{k}\right)=0$$

$$\left[\vec{r}.\left(-4\hat{\imath}+8\hat{\jmath}-16\hat{k}\right)\right]-\left[\left(2\hat{\imath}+3\hat{\jmath}+6\hat{k}\right).\left(-4\hat{\imath}+8\hat{\jmath}-16\hat{k}\right)\right]=0$$

$$\left[\vec{r}.\left(-4\hat{\imath}+8\hat{\jmath}-16\hat{k}\right)\right]-(-8+24-96)=0$$

$$\left[\vec{r}.\left(-4\hat{\imath}+8\hat{\jmath}-16\hat{k}\right)\right]+80=0$$

$$\vec{r}.\left(-4\hat{\imath}+8\hat{\jmath}-16\hat{k}\right)=-80$$

$$\div -4$$

$$\vec{r}.\left(\hat{\imath}-2\hat{\jmath}+4\hat{k}\right)=20$$

Cartesian equation: *Let* $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$

$$\left(x\vec{\imath}+y\vec{\jmath}+z\vec{k}\right).\left(\hat{\imath}-2\hat{\jmath}+4\hat{k}\right)=20$$

$$x-2y+4z-20=0$$

$$x-2y+4z=20$$

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2. *Find the parametric form of vector equation, and Cartesian equations of the plane passing through the points* $(2, 2, 1), (9, 3, 6)$ *and perpendicular to the plane* $2x + 6y + 6z = 9$

Vector equation : $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$

Here $\vec{a} = 2\hat{\imath} + 2\hat{\jmath} + \hat{k}$, $\vec{b} = 9\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$ *and* $\vec{c} = 2\hat{\imath} + 6\hat{\jmath} + 6\hat{k}$

$\vec{b} - \vec{a} = 9\hat{\imath} + 3\hat{\jmath} + 6\hat{k} - 2\hat{\imath} - 2\hat{\jmath} - \hat{k} \Rightarrow \vec{b} - \vec{a} = 7\hat{\imath} + \hat{\jmath} + 5\hat{k}$

$\vec{r} = 2\hat{\imath} + 2\hat{\jmath} + \hat{k} + s(7\hat{\imath} + \hat{\jmath} + 5\hat{k}) + t(2\hat{\imath} + 6\hat{\jmath} + 6\hat{k})$

Cartesian equation :

$(x_1, y_1, z_1) = (2, 2, 1); (x_2, y_2, z_2) = (9, 3, 6);\ (c_1, c_2, c_3) = (2, 6, 6)$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} \overset{+}{x - 2} & \overset{-}{y - 2} & \overset{+}{z - 1} \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix} = 0$$

$(x - 2)(6 - 30) - (y - 2)(42 - 10) + (z - 1)(42 - 2) = 0$

$(x - 2)(-24) - (y - 2)(32) + (z - 1)(40) = 0$

$-24x + 48 - 32y + 64 + 40z - 40 = 0$

$-24x - 32y + 40z + 72 = 0$

$\div -8$

$3x + 4y - 5z - 9 = 0$

$\therefore$ *The parametric form of vector equation is* $\vec{r}.(3\hat{\imath} + 4\hat{\jmath} - 5\hat{k}) = 9$

3. *Find parametric form of vector equation and Cartesian equations of the plane passing through the points* $(2, 2, 1), (1, -2, 3)$ *and parallel to the straight line passing through the points* $(2, 1, -3)$ *and* $(-1, 5, -8)$.

Vector equation : $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$

Here $\vec{a} = 2\hat{\imath} + 2\hat{\jmath} + \hat{k}$, $\vec{b} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$

$\vec{b} - \vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k} - 2\hat{\imath} - 2\hat{\jmath} - \hat{k} \Rightarrow \vec{b} - \vec{a} = -\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$

The straight line passing through the points $(\underset{x_1}{2}, \underset{y_1}{1}, \underset{z_1}{-3})$ *and* $(\underset{x_2}{-1}, \underset{y_2}{5}, \underset{z_2}{-8})$ *is*

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\frac{x - 2}{-1 - 2} = \frac{y - 1}{5 - 1} = \frac{z + 3}{-8 + 3} \Rightarrow \frac{x - 2}{-3} = \frac{y - 1}{4} = \frac{z + 3}{-5}$$

$\vec{c} = -3\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$

$\vec{r} = 2\hat{\imath} + 2\hat{\jmath} + \hat{k} + s(-\hat{\imath} - 4\hat{\jmath} + 2\hat{k}) + t(-3\hat{\imath} + 4\hat{\jmath} - 5\hat{k})$

Cartesian equation :

$$(x_1, y_1, z_1) = (2,2,1); (x_2, y_2, z_2) = (1,-2,3);\ (c_1, c_2, c_3) = (-3,4,-5)$$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} \overset{+}{x-2} & \overset{-}{y-2} & \overset{+}{z-1} \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix} = 0$$

$$(x-2)(20-8) - (y-2)(5+6) + (z-1)(-4-12) = 0$$

$$(x-2)(12) - (y-2)(11) + (z-1)(-16) = 0$$

$$12x - 24 - 11y + 22 - 16z + 16 = 0$$

$$12x - 11y - 16z + 14 = 0$$

4. *Find the non − parametric form of vector equation of the plane passing through the point $(1,-2,4)$ and perpendicular to the plane $x + 2y - 3z = 11$ and parallel to the line $\frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$*

Non – parametric Vector equation : $(\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$

Here $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 4\hat{k}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ *and* $\vec{c} = 3\hat{\imath} - \hat{\jmath} + \hat{k}$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix} = \hat{\imath}(2-3) - \hat{\jmath}(1+9) + \hat{k}(-1-6)$$

$$\vec{b} \times \vec{c} = -\hat{\imath} - 10\hat{\jmath} - 7\hat{k}$$

$$\left(\vec{r} - (\hat{\imath} - 2\hat{\jmath} + 4\hat{k})\right).(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k}) = 0$$

$$[\vec{r}.(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k})] - [(\hat{\imath} - 2\hat{\jmath} + 4\hat{k}).(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k})] = 0$$

$$[\vec{r}.(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k})] - (-1 + 20 - 28) = 0$$

$$[\vec{r}.(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k})] + 9 = 0$$

$$\vec{r}.(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k}) = -9$$

$$\times\ (-)$$

$$\vec{r}.(\hat{\imath} + 10\hat{\jmath} + 7\hat{k}) = 9$$

Cartesian equation: *Let* $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$(x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(\hat{\imath} + 10\hat{\jmath} + 7\hat{k}) = 9$$

$$x + 10y + 7z = 9$$

$$x + 10y + 7z - 9 = 0$$

5. *Find the parametric form of vector equation, and Cartesian equations of the plane containing the line $\vec{r} = (\hat{\imath} - \hat{\jmath} + 3\hat{k}) + t(2\hat{\imath} - \hat{\jmath} + 4\hat{k})$ and perpendicular to plane $\vec{r}.(\hat{\imath} + 2\hat{\jmath} + \hat{k}) = 8$.*

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Vector equation : $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$Here\ \vec{a} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$, $\vec{b} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$ and $\vec{c} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$

$$\vec{r} = \hat{\imath} - \hat{\jmath} + 3\hat{k} + s(2\hat{\imath} - \hat{\jmath} + 4\hat{k}) + t(\hat{\imath} + 2\hat{\jmath} + \hat{k})$$

Cartesian equation :

$(x_1, y_1, z_1) = (1, -1, 3); (b_1, b_2, b_3) = (2, -1, 4); (c_1, c_2, c_3) = (1, 2, 1)$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} \overset{+}{x - 1} & \overset{-}{y + 1} & \overset{+}{z - 3} \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$$(x - 1)(-1 - 8) - (y + 1)(2 - 4) + (z - 3)(4 + 1) = 0$$

$$(x - 1)(-9) - (y + 1)(-2) + (z - 3)(5) = 0$$

$$-9x + 9 + 2y + 2 + 5z - 15 = 0$$

$$-9x + 2y + 5z - 4 = 0$$

$$9x - 2y - 5z + 4 = 0$$

6. *Find the parametric vector, non − parametric vector and Cartesian form of the equations of the plane passing through the points* (3, 6, −2), (−1, −2, 6), *and* (6, −4, −2)

Vector equation: $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a})$

$$\vec{a} = 3\hat{\imath} + 6\hat{\jmath} - 2\hat{k},\ \vec{b} = -\hat{\imath} - 2\hat{\jmath} + 6\hat{k}\ \ and\ \vec{c} = 6\hat{\imath} - 4\hat{\jmath} - 2\hat{k}$$

$$\vec{b} - \vec{a} = -\hat{\imath} - 2\hat{\jmath} + 6\hat{k} - 3\hat{\imath} - 6\hat{\jmath} + 2\hat{k}$$

$$\vec{b} - \vec{a} = -4\hat{\imath} - 8\hat{\jmath} + 8\hat{k}$$

$$\vec{c} - \vec{a} = 6\hat{\imath} - 4\hat{\jmath} - 2\hat{k} - 3\hat{\imath} - 6\hat{\jmath} + 2\hat{k}$$

$$\vec{c} - \vec{a} = 3\hat{\imath} - 10\hat{\jmath}$$

$$\vec{r} = 3\hat{\imath} + 6\hat{\jmath} - 2\hat{k} + s(-4\hat{\imath} - 8\hat{\jmath} + 8\hat{k}) + t(3\hat{\imath} - 10\hat{\jmath})$$

Non – parametric Vector equation: $(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$

$$(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -4 & -8 & 8 \\ 3 & -10 & 0 \end{vmatrix} = \hat{\imath}(0 + 80) - \hat{\jmath}(0 - 24) + \hat{k}\,(40 + 24)$$

$$(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = 80\hat{\imath} + 24\hat{\jmath} + 64\hat{k}$$

$$[\vec{r} - (3\hat{\imath} + 6\hat{\jmath} - 2\hat{k})].(80\hat{\imath} + 24\hat{\jmath} + 64\hat{k}) = 0$$

$$\vec{r}.(80\hat{\imath} + 24\hat{\jmath} + 64\hat{k}) - (3\hat{\imath} + 6\hat{\jmath} - 2\hat{k}).(80\hat{\imath} + 24\hat{\jmath} + 64\hat{k}) = 0$$

$$\vec{r}.(80\hat{\imath} + 24\hat{\jmath} + 64\hat{k}) - (240 + 144 - 128) = 0$$

$$\vec{r}.(80\hat{\imath} + 24\hat{\jmath} + 64\hat{k}) - 256 = 0 \Rightarrow \vec{r}.(80\hat{\imath} + 24\hat{\jmath} + 64\hat{k}) = 256$$

$\div 8$

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$$\vec{r}.(10\hat{\imath}+3\hat{\jmath}+8\hat{k})=32$$

$$(x\hat{\imath}+y\hat{\jmath}+z\hat{k}).(10\hat{\imath}+3\hat{\jmath}+8\hat{k})=32$$

$$10x+3y+8z=32$$

7. Find the no − parametric form of vector equation, and Cartesian equations of the plane
$\boldsymbol{\vec{r}=(6\hat{\imath}-\hat{\jmath}+\hat{k})+s(-\hat{\imath}+2\hat{\jmath}+\hat{k})+t(-5\hat{\imath}-4\hat{\jmath}-5\hat{k})}$.

Non – parametric Vector equation : $(\vec{r}-\vec{a}).(\vec{b}\times\vec{c})=0$

Here $\vec{a}=6\hat{\imath}-\hat{\jmath}+\hat{k}$, $\vec{b}=-\hat{\imath}+2\hat{\jmath}+\hat{k}$ *and* $\vec{c}=-5\hat{\imath}-4\hat{\jmath}-5\hat{k}$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ -1 & 2 & 1\\ -5 & -4 & -5\end{vmatrix}=\hat{\imath}(-10+4)-\hat{\jmath}(5+5)+\hat{k}\,(4+10)$$

$$\vec{b}\times\vec{c}=-6\hat{\imath}-10\hat{\jmath}+14\hat{k}$$

$$\left(\vec{r}-(6\hat{\imath}-\hat{\jmath}+\hat{k})\right).(-6\hat{\imath}-10\hat{\jmath}+14\hat{k})=0$$

$$[\vec{r}.(-6\hat{\imath}-10\hat{\jmath}+14\hat{k})]-[(6\hat{\imath}-\hat{\jmath}+\hat{k}).(-6\hat{\imath}-10\hat{\jmath}+14\hat{k})]=0$$

$$[\vec{r}.(-6\hat{\imath}-10\hat{\jmath}+14\hat{k})]-(-36+10+14)=0$$

$$[\vec{r}.(-6\hat{\imath}-10\hat{\jmath}+14\hat{k})]+12=0$$

$$\div\ (-2)$$

$$[\vec{r}.(3\hat{\imath}+5\hat{\jmath}-7\hat{k})]-6=0$$

Cartesian equation: *Let* $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$

$$(x\vec{\imath}+y\vec{\jmath}+z\vec{k}).(3\hat{\imath}+5\hat{\jmath}-7\hat{k})-6=0$$

$$3x+5y-7z-6=0$$

Excercise 6.8

Example 6.45: Verify whether the line $\frac{x-3}{-4}=\frac{y-4}{-7}=\frac{z+3}{12}$ ***lies in the plane*** $5x-y+z=8$.

Let $(x_1,y_1,z_1)=(3,4,-3)\quad (a,b,c)=(-4,-7,12)\quad (A,B,C)=(5,-1,1)$

Condition for a line to lie in a plane $aA+bB+cC=0$

$$aA+bB+cC=(-4)(5)+(-7)(-1)+(12)(1)$$
$$=-20+7+12=-1\neq 0$$

Hence, the given line does not lie in the plane.

3. ***If the straight lines*** $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{m^2}$ ***and*** $\frac{x-3}{1}=\frac{y-2}{m^2}=\frac{z-1}{2}$ ***are coplanar, find the distinct real values of m.***

Let $\vec{a}=\hat{\imath}+2\hat{\jmath}+3\hat{k},\ \vec{b}=\hat{\imath}+2\hat{\jmath}+m^2\hat{k},\ \vec{c}=\hat{\imath}+2\hat{\jmath}+\hat{k},\ \vec{d}=\hat{\imath}+m^2\hat{\jmath}+2\hat{k}$

The two given lines are co − planar

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=0$$

$$\vec{c}-\vec{a}=\hat{\imath}+2\hat{\jmath}+\hat{k}-(\hat{\imath}+2\hat{\jmath}+3\hat{k})$$
$$=\hat{\imath}+2\hat{\jmath}+\hat{k}-\hat{\imath}-2\hat{\jmath}-3\hat{k}$$

$$\vec{c}-\vec{a}=-2\hat{k}$$

$$\vec{b}\times\vec{d}=\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\1&2&m^2\\1&m^2&2\end{vmatrix}=\hat{\imath}(4-m^4)-\hat{\jmath}(2-m^2)+\hat{k}\,(m^2-2)$$

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=0$$

$$(-2\hat{k}).\left[\hat{\imath}(4-m^4)-\hat{\jmath}(2-m^2)+\hat{k}\,(m^2-2)\right]=0$$

$$-2(m^2-2)=0\implies m^2-2=0$$
$$m^2=2\implies m=\pm\sqrt{2}$$

4. ***If the straight lines*** $\frac{x-1}{2}=\frac{y+1}{\lambda}=\frac{z}{2}$ ***and*** $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{\lambda}$ ***are coplanar, find λ and equations of the planes containing these two lines.***

Let $\vec{a}=\hat{\imath}-\hat{\jmath},\ \vec{b}=2\hat{\imath}+\lambda\hat{\jmath}+2\hat{k},\ \vec{c}=-\hat{\imath}-\hat{\jmath},\ \vec{d}=5\hat{\imath}+2\hat{\jmath}+\lambda\hat{k}$

The two given lines are co − planar

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=0$$

$$\vec{c}-\vec{a}=-\hat{\imath}-\hat{\jmath}-(\hat{\imath}-\hat{\jmath})=-\hat{\imath}-\hat{\jmath}-\hat{\imath}+\hat{\jmath}$$

$$\boxed{\vec{c}-\vec{a}=-2\hat{\imath}}$$

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$$\vec{b}\times\vec{d}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & \lambda & 2\\ 5 & 2 & \lambda\end{vmatrix}=\hat{\imath}(\lambda^2-4)-\hat{\jmath}(2\lambda-10)+\hat{k}\,(4-5\lambda)$$

$$(\vec{c}-\vec{a}).\left(\vec{b}\times\vec{d}\right)=0$$

$$(-2\hat{\imath}).\left[\hat{\imath}(\lambda^2-4)-\hat{\jmath}(2\lambda-10)+\hat{k}\,(4-5\lambda)\right]=0$$

$$-2(\lambda^2-4)=0\Longrightarrow\ \lambda^2-4=0$$

$$\lambda^2=4\Longrightarrow\ \lambda=\pm\sqrt{4}$$

$$\lambda=\pm2$$

Cartesian equation :

$(x_1, y_1, z_1)=(1,-1,0)$; $(b_1, b_2, b_3)=(2,2,2)$; $(d_1, d_2, d_3)=(5,2,2)$

$$\begin{vmatrix}x-x_1 & y-y_1 & z-z_1\\ b_1 & b_2 & b_3\\ d_1 & d_2 & d_3\end{vmatrix}=0 \quad\Longrightarrow \begin{vmatrix}\overset{+}{x-1} & \overset{-}{y+1} & \overset{+}{z}\\ 2 & 2 & 2\\ 5 & 2 & 2\end{vmatrix}=0$$

$$(x-1)(4-4)-(y+1)(4-10)+(z)(4-10)=0$$

$$(x-1)(0)-(y+1)(-6)+z\,(-6)=0$$

$$6(y+1)-6z=0\Longrightarrow\ 6(y+1-z)=0$$

$$y-z+1=0\ for\ \lambda=2$$

$$y+z+1=0\ for\ \lambda=-2$$

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Excercise 6.9

Example 6.47: Find the acute angle between the planes $\vec{r}.\left(2\hat{\imath}+2\hat{\jmath}+2\hat{k}\right)=11$ ***and*** $4x-2y+2z=15$.

$$\vec{r}.\left(2\hat{\imath}+2\hat{\jmath}+2\hat{k}\right)=11 \text{ and } 4x-2y+2z=15$$

$$\vec{n}_1=2\hat{\imath}+2\hat{\jmath}+2\hat{k} \text{ and } \vec{n}_2=4\hat{\imath}-2\hat{\jmath}+2\hat{k}$$

$$\cos\theta=\frac{|\vec{n}_1.\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$$

$$\cos\theta=\frac{\left|\left(2\hat{\imath}+2\hat{\jmath}+2\hat{k}\right).\left(4\hat{\imath}-2\hat{\jmath}+2\hat{k}\right)\right|}{\sqrt{2^2+2^2+2^2}\sqrt{4^2+(-2)^2+2^2}}$$

$$\cos\theta=\frac{8-4+4}{\sqrt{4+4+4}\sqrt{16+4+4}} \Rightarrow \cos\theta=\frac{8}{\sqrt{12}\sqrt{24}}$$

$$\cos\theta=\frac{8}{2\sqrt{3}\times2\sqrt{6}} \Rightarrow \cos\theta=\frac{2}{\sqrt{18}} \Rightarrow \cos\theta=\frac{2}{3\sqrt{2}}$$

$$\cos\theta=\frac{\sqrt{2}\times\sqrt{2}}{3\sqrt{2}} \Rightarrow \cos\theta=\frac{\sqrt{2}}{3} \Rightarrow \theta=\cos^{-1}\frac{\sqrt{2}}{3}$$

Example 6.48: Find the angle between the straight line $\vec{r}=\left(2\hat{\imath}+3\hat{\jmath}+\hat{k}\right)+t\left(\hat{\imath}-\hat{\jmath}+\hat{k}\right)$ ***and the plane*** $2x-y+z=5$.

Let $\vec{b}=\hat{\imath}-\hat{\jmath}+\hat{k}$ *and* $\vec{n}=2\hat{\imath}-\hat{\jmath}+\hat{k}$

Angle between line and plane is $\sin\theta=\frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}$

$$\sin\theta=\frac{\left(\hat{\imath}-\hat{\jmath}+\hat{k}\right).\left(2\hat{\imath}-\hat{\jmath}+\hat{k}\right)}{\sqrt{1^2+(-1)^2+1^2}\sqrt{2^2+(-1)^2+1^2}}$$

$$\sin\theta=\frac{2+1+1}{\sqrt{1+1+1}\sqrt{4+1+1}}$$

$$\sin\theta=\frac{4}{\sqrt{3}\sqrt{6}} \Rightarrow \sin\theta=\frac{4}{\sqrt{18}} \Rightarrow \sin\theta=\frac{4}{\sqrt{2\times3\times3}}$$

$$\sin\theta=\frac{4}{3\sqrt{2}} \Rightarrow \sin\theta=\frac{4\sqrt{2}}{3\sqrt{2}\times\sqrt{2}} \Rightarrow \sin\theta=\frac{4\sqrt{2}}{3\times2}$$

$$\sin\theta=\frac{2\sqrt{2}}{3} \Rightarrow \theta=\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

Example 6.49: Find the distance from the point $(2,5,-3)$ ***to the plan*** $\vec{r}.\left(6\hat{\imath}-3\hat{\jmath}+2\hat{k}\right)=5$

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$$\vec{r}.\left(6\hat{\imath}-3\hat{\jmath}+2\hat{k}\right)=5 \quad where\ \vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$$

$$\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right).\left(6\hat{\imath}-3\hat{\jmath}+2\hat{k}\right)=5 \Rightarrow 6x-3y+2z=5$$

$$6x-3y+2z-5=0$$

$$The\ distance\ from\ the\ point\ to\ the\ plane=\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$

$$a=6, b=-3, c=2, d=-5$$

$$(x_1,y_1,z_1)=(2,5,-3)$$

$$=\left|\frac{(6)(2)+(-3)(5)+(2)(-3)-5}{\sqrt{6^2+(-3)^2+2^2}}\right|=\left|\frac{12-15-6-5}{\sqrt{36+9+4}}\right|$$

$$=\left|\frac{12-26}{\sqrt{49}}\right|=\left|\frac{-14}{7}\right|=2\ units$$

***Example 6.50**: Find the distance of a point $(5,-5,-10)$ from the point of intersection of a straight line passing through the points $A(4,1,2)$ and $B(7,5,4)$ with the plane $x-y+z=5$.*

The cartesian equation of the straight line joining A and B is

Here $(x_1, y_1, z_1)=(4,1,2)$ *and* $(x_2, y_2, z_2)=(7,5,4)$.

***Cartesian equation**:*

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \Rightarrow \frac{x-4}{3}=\frac{y-1}{4}=\frac{z-2}{2}$$

$$\frac{x-4}{3}=\frac{y-1}{4}=\frac{z-2}{2}=t \Rightarrow \frac{x-4}{3}=t, \frac{y-1}{4}=t, \frac{z-2}{2}=t$$

$$x=3t+4, y=4t+1, z=2t+2 \Rightarrow (3t+4, 4t+1, 2t+2)$$

$\therefore$ *An arbitrary point on the straight line is of the form*

$(3t+4, 4t+1, 2t+2)$

To find the point of intersection of the straight line and the plane

$(3t+4, 4t+1, 2t+2)$ *in* $x-y+z=5$

$$3t+4-(4t+1)+2t+2=5$$

$$3t+4-4t-1+2t+2=5$$

$$t+5-5=0 \Rightarrow t=0$$

sub $t=0$ *in* $(3t+4, 4t+1, 2t+2) \Rightarrow (3\times 0+4, 4\times 0+1, 2\times 0+2)$

$\therefore$ *The point of intersection of the straight line is* $(4,1,2)$.

The distance between the two points $(\underset{x_1}{4}, \underset{y_1}{1}, \underset{z_1}{2})$ *and* $(\underset{x_2}{5}, \underset{y_2}{-5}, \underset{z_2}{-10})$ *is*

$$=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$$=\sqrt{(5-4)^2+(-5-1)^2+(-10-2)^2}$$

$$= \sqrt{(1)^2 + (-6)^2 + (-12)^2} = \sqrt{1 + 36 + 144}$$

$$= \sqrt{181}\ units$$

Example 6.51: Find the distance between the parallel planes $x + 2y - 2z + 1 = 0$ and $2x + 4y - 4z + 5 = 0$.

The distance between the parallel planes $x + 2y - 2z + 1 = 0$ *and*

$$2x + 4y - 4z + 5 = 0$$

$$\div 2$$

$$x + 2y - 2z + \frac{5}{2} = 0$$

$$a = 1,\ b = 2,\ c = -2,\ d_1 = 1,\ d_2 = \frac{5}{2}$$

$$Distance = \left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\frac{1 - \frac{5}{2}}{\sqrt{1^2 + 2^2 + (-2)^2}}\right|$$

$$= \left|\frac{\frac{2-5}{2}}{\sqrt{1 + 4 + 4}}\right| = \left|\frac{\frac{3}{2}}{\sqrt{9}}\right| = \left|\frac{\frac{3}{2}}{3}\right| = \left|\frac{3}{2} \times \frac{1}{3}\right|$$

$$Distance = \frac{1}{2}$$

Example 6.52: Find the distance between the planes $\vec{r}.(2\hat{i} - \hat{j} - 2\hat{k}) = 6$ and $\vec{r}.(6\hat{i} - 3\hat{j} - 6\hat{k}) = 27$.

$$\vec{r}.(2\hat{i} - \hat{j} - 2\hat{k}) = 6\ \ where\ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$(x\hat{i} + y\hat{j} + z\hat{k}).(2\hat{i} - \hat{j} - 2\hat{k}) = 6 \Longrightarrow 2x - y - 2z = 6$$

$$\vec{r}.(6\hat{i} - 3\hat{j} - 6\hat{k}) = 27\ \ where\ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$(x\hat{i} + y\hat{j} + z\hat{k}).(6\hat{i} - 3\hat{j} - 6\hat{k}) = 27 \Longrightarrow 6x - 3y - 6z = 27$$

$$\div 3$$

$$2x - y - 2z = 9$$

The Cartesian equation of the planes are $2x - y - 2z = 3$ *and* $2x - y - 2z = 9$

$$a = 2,\ b = -1,\ c = -2,\ d_1 = 6,\ d_2 = 9$$

$$Distance = \left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\frac{6 - 9}{\sqrt{2^2 + (-1)^2 + (-2)^2}}\right|$$

$$= \left|\frac{-3}{\sqrt{4 + 1 + 4}}\right| = \left|\frac{-3}{\sqrt{9}}\right| = \left|\frac{-3}{3}\right| = 1$$

3. *Find the angle between the line $\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + 2\hat{j} - 2\hat{k})$ and the plane $\vec{r}.(6\hat{i} + 3\hat{j} + 2\hat{k}) = 8$.*

Let $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ *and* $\vec{n} = 6\hat{i} + 3\hat{j} + 2\hat{k}$

Angle between line and plane is

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$$\sin\theta = \frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}$$

$$\sin\theta = \frac{(\hat{\imath}+2\hat{\jmath}-2\hat{k}).(6\hat{\imath}+3\hat{\jmath}+2\hat{k})}{\sqrt{1^2+2^2+(-2)^2}\sqrt{6^2+3^2+2^2}}$$

$$\sin\theta = \frac{6+6-4}{\sqrt{1+4+4}\sqrt{36+9+4}}$$

$$\sin\theta = \frac{8}{\sqrt{9}\sqrt{49}} \Longrightarrow \sin\theta = \frac{8}{3\times 7} \Longrightarrow \sin\theta = \frac{8}{21}$$

$$\boxed{\theta = sin^{-1}\left(\frac{8}{21}\right)}$$

4. Find the angle between the planes $\vec{r}.(\hat{\imath}+\hat{\jmath}-2\hat{k}) = 3$ and $2x-2y+z=2$.

$$\vec{r}.(\hat{\imath}+\hat{\jmath}-2\hat{k}) = 3 \text{ and } 2x-2y+z=2$$

$$\vec{n}_1 = \hat{\imath}+\hat{\jmath}-2\hat{k} \text{ and } \vec{n}_2 = 2\hat{\imath}-2\hat{\jmath}+\hat{k}$$

$$\cos\theta = \frac{|\vec{n}_1.\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$$

$$\cos\theta = \frac{|(\hat{\imath}+\hat{\jmath}-2\hat{k}).(2\hat{\imath}-2\hat{\jmath}+\hat{k})|}{\sqrt{1^2+1^2+(-2)^2}\sqrt{2^2+(-2)^2+1^2}}$$

$$\cos\theta = \frac{|2-2-2|}{\sqrt{1+1+4}\sqrt{4+4+1}}$$

$$\cos\theta = \frac{|-2|}{\sqrt{6}\sqrt{9}} \Longrightarrow \cos\theta = \frac{2}{3\sqrt{6}} \Longrightarrow \boxed{\theta = cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)}$$

5. Find the equation of the plane which passes through the point $(3,4,-1)$ and is parallel to the plane $2x-3y+5z+7=0$. Also, find the distance between the twoplanes.

Equation of the given plane is $2x-3y+5z+7=0$

Equation of the plane parallel to the given plane is $2x-3y+5z+k=0$

Since this plane passes through the point $(3,4,-1)$.

$$2(3)-3(4)+5(-1)+k=0 \Longrightarrow 6-12-5+k=0$$

$$-11+k=0 \Longrightarrow k=11$$

$$k=11 \text{ in } 2x-3y+5z+k=0$$

Required plane is $2x-3y+5z+11=0$

$Equation\ of\ the\ plane\ are\ 2x - 3y + 5z + 7 = 0\ and\ 2x - 3y + 5z + 11 = 0$

$$Distance\ between\ two\ parallel\ planes\quad d = \left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$a = 2\ , b = -3, c = 5, d_1 = 7\ , d_2 = 11$$

$$d = \left|\frac{7 - 11}{\sqrt{2^2 + (-3)^2 + 5^2}}\right| = \left|\frac{-4}{\sqrt{4 + 9 + 25}}\right| = \left|\frac{-4}{\sqrt{38}}\right|$$

$$d = \frac{4}{\sqrt{38}}\ units$$

6. *Find the length of the perpendicular from the point* $(1, -2, 3)$ *to the plane* $x - y + z = 5$.

$Length\ of\ perpendicular\ from\ (x_1, y_1, z_1)\ to\ the\ plane\ ax + by + cz - d = 0$

$$is\ \left|\frac{ax_1 + by_1 + cz_1 - d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$\therefore Length\ of\ perpendicular\ from\ \underset{(x_1, y_1, z_1)}{(1, -2, 3)}\ to\ the\ plane\ x - y + z - 5 = 0\ is$

$$a = 1\ , b = -1, c = 1, d = -5$$

$$= \left|\frac{1(1) - 1(-2) + 1(3) - 5}{\sqrt{1^2 + (-1)^2 + 1^2}}\right|$$

$$= \left|\frac{1(1) - 1(-2) + 1(3) - 5}{\sqrt{1^2 + (-1)^2 + 1^2}}\right| = \left|\frac{1 + 2 + 3 - 5}{\sqrt{1 + 1 + 1}}\right| = \left|\frac{1}{\sqrt{3}}\right| = \frac{1}{\sqrt{3}}\ units$$

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